NAVAL POSTGRADUATE SCHOOL

MONTEREY, CALIFORNIA

THESIS

BUYING HEARTS AND MINDS: MODELING POPULAR SUPPORT DURING AN INSURGENCY VIA A SEQUENTIAL VOTE-BUYING GAME

by

Baris Yildiz

June 2011

Thesis Advisor:	Michael Atkinson
Second Reader:	Roberto Szechtman

Approved for public release; distribution is unlimited

THIS PAGE INTENTIONALLY LEFT BLANK

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704–0188

1. REPORT DATE *(DD–MM–YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From — To)*
8–6–2011	Master's Thesis	2009-09-28—2011-06-17

4. TITLE AND SUBTITLE

Buying Hearts and Minds: Modeling Popular Support During an Insurgency Via a Sequential Vote-Buying Game

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

Baris Yildiz

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Postgraduate School
Monterey, CA 93943

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)

N/A

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited

13. SUPPLEMENTARY NOTES

The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the Turkish Government. **IRB Protocol Number: N/A**

14. ABSTRACT

The population plays a crucial role in the outcome of an insurgency. The government needs intelligence from the population to effectively target and defeat the insurgents. In this thesis, we adapt a vote-buying model from political science to the insurgency context to analyze the level of intelligence the population will provide to the government. The model is a two-player sequential game in which both the government and insurgents can "pay" individuals for their support. These payments can take the form of direct bribes or the provision of benefits, such as building schools and roads. In the model, an individual supports the government by providing it with intelligence. We specify the optimal payment strategy for the insurgents and the government and determine how much support the government will receive. In an extension to the base model we allow the insurgents to use coercion as a means to deter individuals from supporting the government. Our analysis illustrates that coercion can be an effective tool for the insurgents in some situations but may backfire in others.

15. SUBJECT TERMS

Counterinsurgency, Game Theory, Vote-buying, Coercion

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			19b. TELEPHONE NUMBER *(include area code)*
Unclassified	Unclassified	Unclassified	UU	119	

NSN 7540-01-280-5500

Standard Form 298 (Rev. 8–98)
Prescribed by ANSI Std. Z39.18

THIS PAGE INTENTIONALLY LEFT BLANK

Approved for public release; distribution is unlimited

BUYING HEARTS AND MINDS: MODELING POPULAR SUPPORT DURING AN INSURGENCY VIA A SEQUENTIAL VOTE-BUYING GAME

Baris Yildiz
First Lieutenant, Turkish Army
B.S., Systems Engineering, Turkish Military Academy 2002

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
June 2011

Author: Baris Yildiz

Approved by: Michael Atkinson
Thesis Advisor

Roberto Szechtman
Second Reader

Robert F. Dell
Chair, Department of Operations Research

THIS PAGE INTENTIONALLY LEFT BLANK

ABSTRACT

The population plays a crucial role in the outcome of an insurgency. The government needs intelligence from the population to effectively target and defeat the insurgents. In this thesis, we adapt a vote-buying model from political science to the insurgency context to analyze the level of intelligence the population will provide to the government. The model is a two-player sequential game in which both the government and insurgents can "pay" individuals for their support. These payments can take the form of direct bribes or the provision of benefits, such as building schools and roads. In the model, an individual supports the government by providing it with intelligence. We specify the optimal payment strategy for the insurgents and the government and determine how much support the government will receive. In an extension to the base model we allow the insurgents to use coercion as a means to deter individuals from supporting the government. Our analysis illustrates that coercion can be an effective tool for the insurgents in some situations but may backfire in others.

THIS PAGE INTENTIONALLY LEFT BLANK

Table of Contents

List of Figures

THIS PAGE INTENTIONALLY LEFT BLANK

List of Tables

THIS PAGE INTENTIONALLY LEFT BLANK

Executive Summary

Recent insurgencies illustrate the crucial importance of incorporating the population into an effective counterinsurgency strategy. The government needs intelligence from the population to target the insurgents with force. The insurgents will take actions to prevent the flow of intelligence from the population to the government. These actions may include the provision of benefits or threats of violence. In this thesis, we analyze the battle for the hearts and minds of the population in an insurgency context. We formulate a mathematical model based on a vote-buying model from political science. There is a natural connection between these two situations, as the government and insurgents vie for popular support just as two candidates would in a standard election. The model is a two-player game where the government first makes an offer to individuals in the population in return for intelligence. The insurgents observe the actions taken by the government and make a counter-offer. These offers might be as simple and crude as a monetary bribe, but they could be more complex. The insurgents or government might offer benefits to the population including providing education, health care, security, and improved infrastructures. Individuals consider the two offers and then accept the offer that provides them with more value, while accounting for their innate preferences. Each individual may have ideological tendencies that more naturally align with either the insurgents or the government. We also incorporate the impact of coercion by the insurgents into the model. Finally, we examine how the results differ when the population is a heterogeneous mix of individuals compared to a population of only a small number of powerful tribes.

Our study primarily focuses on two issues: the minimum cost (or strength) required for the government to "win" the popular support component of the counterinsurgency campaign, and the level of support the government will receive if it wins. We also investigate situations where coercion is an effective tool for the insurgents and situations where coercion may actually help the government obtain more intelligence. Below we list the primary insights from the analysis of the model.

If the government wins, it will receive more intelligence than is required to achieve victory. This occurs because the insurgents are strategic and can easily undermine the government's support if it is only the minimum necessary for government victory. Examining the battle for the hearts and minds from a decision analysis approach will lead to a different and incorrect conclusion. As expected, the government is more likely to win if the intelligence requirement is low or the insurgents are weak. The level of support obtained by the government increases

with the strength of the insurgents. This is another conclusion from the strategic analysis of the problem. A strong insurgency will be able to poach supporters from the government, and so the government needs to form a large base of support to prevent this. The government will also gain more supporters if the population has weak preferences for the government or insurgents. In this case it is less costly for the government to convince pro-insurgent individuals to provide them with intelligence. The key insight about coercion is that it can be a double-edged sword for the insurgents. In some situations it is an effective tool that silences government supporters. However in other situations the insurgents may cause pro-insurgent individuals to provide the government with intelligence because of their brazen use of violence. If the government wins in the face of coercion, their level of support will increase. One of the main contributions of this work is defining a weight to individuals in the population. This value can represent the clout or influence an individual has in the population. Our results are sensitive to the specific form of the weight function. Finally we examine a population comprised of a small number of tribes, which is much more difficult to analyze then a heterogeneous population. We formulate an optimization algorithm to solve for the optimal strategy of the government. The results for a tribal population can differ significantly from the results corresponding to a population of many independent and unique individuals. These last several observations suggest that care should be taken when estimating the underlying structure of the society.

Acknowledgements

It is not possible to put into words my gratitude to Professor Michael Atkinson. I appreciate all of his guidance, and words cannot express how grateful I am for his unwavering patience and limitless support that turned this journey into an exceptional learning opportunity. It would have been impossible to complete this research without his endless support.

I also want to express my thanks to Professor Roberto Szechtman for his commitment to ensuring that this thesis pushed the boundaries of academic research.

I want to express special thanks to my wife, Ozlem, for her unconditional love and unwavering support, on which I stood in comfort throughout this arduous journey. I am also obligated to express my warmest thanks to my daughter Elif, who wiped away the exhaustion from the long work days with the warmest little hugs in the world.

Finally, I would like to offer my profound thanks to my country, Turkey, for providing me the great opportunity to attend the Operations Research program at NPS.

THIS PAGE INTENTIONALLY LEFT BLANK

CHAPTER 1: Introduction

The counterinsurgency campaigns in Afghanistan, Iraq, and elsewhere illustrate the difficulty in fighting insurgents and the importance of having an effective counterinsurgency strategy. One crucial way that counterinsurgency operations differ from conventional conflicts is the role of population [1]. The population plays a critical role in insurgency situations: an insurgency cannot sustain itself without popular support. The support may involve active aid to the insurgents: recruits, finances, resources, sanctuary, etc [1]. However, the support to the insurgents may take a more passive form, such as the population not providing the government with intelligence about the insurgents' location or tactics [1]. Mao Tse-tung captures the insurgents' dependence on the population in this quote: "The people are water, the Red Army are fish; without water, the fish will die"[2]. Therefore, the government needs to incorporate the population into their counterinsurgency strategy by focusing on weakening the connections between the insurgents and population and fostering intelligence sources within the population [3]. In this thesis we primarily focus on the latter.

Policymakers recognize the important role that the population plays in gaining vital intelligence for the government [FM 3-24 Counterinsurgency]. In his March 17, 2005, Congressional testimony, Executive Director of Center for Strategic and Budgetary Assessments Andrew F. Krepinevich has stated:

> In conventional warfare, the enemy's military forces are often seen as its center of gravity. . . This is not the case in insurgency warfare, where the population is the center of gravity. Defeating them (insurgents) requires time, both to provide counterinsurgent forces with an understanding of the environment in which the insurgent forces are operating, and to win the hearts and minds of the population, which will produce the human intelligence needed to distinguish the enemy from noncombatants. . . The key to defeating an insurgency is to attack it at the source of its strength: the population. If the counterinsurgent forces can deny the insurgents access to the people, the insurgents become like fish out of water, denied sources of manpower and information.

The most impressive example of the population impacting counterinsurgency operations occurred in Anbar during Operation Iraqi Freedom. During a September 2007 visit to Anbar

Province in western Iraq, President George W. Bush stated, “Anbar is a huge province. It was once written off as lost. It is now one of the safest places in Iraq.” [5]. The reason for this remarkable reversal was that individuals “who once fought side by side with al Qaeda against coalition troops [are] now fighting side by side with coalition troops against al Qaeda” [5].

While there are many factors that determine who “wins” an insurgency conflict, we focus in this thesis on the aspect discussed in the preceding paragraphs: winning the hearts and minds of the population. More specifically we concentrate on the government acquiring enough intelligence so that they can effectively target the insurgents with force. In our model, we assume that if the government obtains enough intelligence from the population then they win the battle for the population, otherwise the insurgents win. The outcome of this component of the conflict then directly influences the larger counterinsurgency campaign. Thus, the core of our model is that the government is going to take actions to increase the amount of intelligence it receives, while the insurgents are a strategic adversary that will counteract the tactics of the government to minimize the intelligence flowing to the government from the population.

We formulate a mathematical model based on a vote-buying model from political science [6]. There is a natural connection between these two situations. In our context, the government and insurgents vie for popular support just as two candidates for president would in a standard election. The model is a two-player game where the government first makes an offer to individuals in the population in return for intelligence. The insurgents observe the actions taken by the government and make a counter-offer. These offers might be as simple and crude as a monetary bribe (and this is how we will denote it throughout the thesis), but they could be more complex. The insurgents or government might offer benefits to the population including providing education, health care, security, and improved infrastructures. Individuals consider the two offers and then accept the offer that provides them with more value, while accounting for their innate preferences. Each individual may have ideological tendencies that more naturally align with either the insurgents or the government. We also incorporate the impact of coercion by the insurgents into the model. Finally, we examine how the results differ when the population is a heterogeneous mix of individuals compared to a population of only a small number of powerful tribes.

One of the main contributions of this thesis is to frame an insurgency situation within a vote-buying framework. After formulating the vote-buying game, we specify the complete equilibrium of the game. As with most attacker-defender type games, the government will provide a

bribe to individuals such that most are equally costly to the insurgents to counter-bribe. To win, the government must obtain support from more individuals than is necessary for the government to achieve victory. This occurs because it prevents the insurgents from being able to mount a winning counter-offer. We incorporate the impact of coercion and study cases where it is an effective tool for the insurgents. However, coercion can be a double-edged sword and there are situations where it backfires on the insurgents and makes it more likely that the government will win. Most of the analysis assumes that the population consists of a continuum of individuals. However to examine the case where the population comprises a handful of tribes we must modify the model, which produces a much more complicated formulation. While we derive limited analytic insights from this discrete model, we do provide a numerical approach to solve it.

The rest of thesis is organized as follows. In Chapter 2, we discuss the literature related to this problem and our modeling approach. The bulk of the thesis appears in Chapter 3, which provides a more thorough explanation of the background and motivation of the model as well as the initial model formulation and its extensions. We illustrate the model and its results in Chapter 4 before concluding in Chapter 5 with a discussion of the results, shortcomings of the model, and potential future work. Technical details supporting the results in Chapter 3 appear in the Appendix.

THIS PAGE INTENTIONALLY LEFT BLANK

CHAPTER 2:
Background Research and Literature Review

We formulate a two-player sequential game to gain insight about the competition between the government and insurgents for the support of the general population. We do this by adapting a vote-buying model from political science by Groseclose and Snyder [6]. Before discussing the vote-buying literature in general in Section 2.3 and the Groseclose and Snyder model in particular in Section 2.4, we discuss two related topics. In Section 2.1, we present an overview of auction literature and in Section 2.2 we discuss Colonel Blotto games. Finally, we discuss counterinsurgency and social dynamics in Sections 2.5 and 2.6, respectively.

2.1 Auction Theory

The original motivation of this thesis was that in a simplified world, the government and insurgents were bidders in an auction and they would bid for the support of individual tribes. Through reading the auction literature, primarily [7], we uncovered the vote-buying literature. This line of works seems a more natural fit than standard auction theory. While vote buying may differ from traditional auctions, it has similarities with nonstandard auctions. Krishna [8] overviews auction theory with an extensive survey of the field. In the literature there are four "standard" auctions studied: the ascending-bid auction (English auction), the descending-bid auction (Dutch auction), the first-price sealed-bid auction and the second-price sealed-bid auction (i.e., the Vickrey auction) [9]. Along with these standard auctions, there are a myriad of other theoretical and practical auction settings investigated by various studies. See Krishna and Morgan [10] and Che and Gale [11, 12] for recent studies that investigate the theory of non-standard auction formats. The two types of auctions that are the most relevant for our purposes are the war of attrition and the all-pay auction. Definitions of these non-standard auctions are given in Krishna and Morgan [10] as follows:

> These auction forms share the common feature that all losing bidders pay exactly their bids and differ only in the amounts paid by the winning bidder. In the [war of attrition auction], the winning bidder pays the second highest bid; whereas in the [all-pay auction], the winner pays his own bid. As such, they are analogous to the standard second-price and first-price sealed bid auctions, respectively.

Models for conflict among animals and insects [13, 14], survival among firms [15] , arms race [16], R&D competitions [17] and lobbying [7, 18, 19] are some of the examples where war of attrition and all-pay auctions are used analyze underlying games [10]. Vote-buying can be modeled as an all-pay auction because all parties have to pay for the vote whether they win or lose [7, 18, 19]. Election manipulation via campaign promises or clientelism resembles a standard first-price auction where the losing parties do not need to pay [7, 20].

2.2 Colonel Blotto Games

The Colonel Blotto game captures strategic situations in which players simultaneously allocate forces to various cells on a battlefield. A side wins a cell if it allocates more force to the cell than its adversary [21]. The battle for popular support of the general population during an insurgency can be can be interpreted as a Blotto Game where the government will receive support from a tribe if they provide that tribe with more utility (e.g., payments, health-care, schools, infrastructure) than the insurgents do.

First proposed by Borel [22] in 1921 and solved for the special case of three battlefields, classic Blotto games are difficult to solve, due to the simultaneous nature of the interaction. Gross and Wagner [23] provide a solution for a finite number of battlefields if the players have symmetric resources. Friedman [24] provides a partial characterization of the solution to the problem for n-battlefields and asymmetric resources. More recently, Roberson [25] has applied the theory of copulas to prove the uniqueness of equilibrium payoffs for finite number of battlefields with players having arbitrary asymmetric resources [26].

Gerchak and Parlar [27] examine the allocation of resources to R&D activities in an uncertain and competitive environment with winner-takes-all payoffs. Instead of the assuming the party that allocates a larger force than the other captures the payoff with probability 1, they investigate the case where the probability of capturing the payoff is a function of the bids of the rivals. They provide an optimal resource allocation between two activities conditional on the competitor's allocation and characterize the Nash and Stackelberg equilibria for that model. Our formulation follows the initial Blotto model in that the outcome for each tribe is binary: one side completely wins support from the tribe. In practice, both the insurgents and the government may receive support from the same tribe. Thus future work will incorporate the ideas of [27] to split the support of the tribes. In the base model, the population is a continuum and there are no tribes, but just individuals. Therefore, this formulation captures in a rough sense a tribe that splits its support between the government and insurgents.

The closest study to our work is done by Powell [28], who examines the strategic allocation of resources across multiple fronts in the context of nonzero sum Blotto games. In his formulation, as in our formulation, the players sequentially select their allocations. This variation makes the problem much more tractable. The defender first places defense assets around targets, and then the attacker chooses how much attack resources to allocate to each target. He shows that unlike many Blotto games which only have very complicated mixed-strategy equilibria, the sequential, nonzero-sum Blotto game has a pure-strategy subgame perfect equilibrium. The defender always plays the same pure strategy, and the attacker's response is generally unique and entails no mixing.

2.3 Vote Buying Models

Our model is most closely related to vote buying model of Groseclose and Snyder (GS) [6]. Groseclose and Snyder presents a model of vote buying in a legislature. They examine the equilibrium coalition size with preferences of the legislators and vote buyers in a sequential game. We dedicate section 2.4 specifically to their study and defer the further discussion of this paper until then. While the Groseclose and Snyder model primarily focuses on a continuum of voters, Banks [29] and Dippel [30] analyze the discrete-voter population and identify how the optimal coalition size varies with the underlying preference parameters. They derive the necessary and sufficient conditions for minimal majority and universal coalitions to form. Their findings are consistent with the original Groseclose and Snyder model. However, the key assumption in [29, 30] is that all voters have equal weight. While that is a natural and reasonable assumption in most cases, it may be a poor assumption when considering tribal societies. Therefore, we consider a population where individuals or tribes have different weights. While the results of the continuous version of this model are quite similar to the Groseclose and Snyder model, the discrete version is much different and more complicated to analyze.

Dekel and coauthors have two papers that examine vote buying scenarios [7, 31]. The work in [7] analyzes simultaneous-move vote buying, where voters care about how they cast their vote and not about the eventual outcome. This is common in legislatures where individuals want to be reelected and thus only care about how their constituents perceive their vote. In their companion paper [31], they compare the vote manipulation process of using campaign promises vs. direct payments. Payments are contingent on the vote, whereas campaign promises are contingent on the outcome of the election. They find that direct vote buying provides far less total social benefits than campaign promises. It would be interesting to incorporate a concept similar to

campaign promises in [31] in future extensions of the model in this thesis.

Morgan and Várdy [32] investigate the ease with which various voting bodies can be bought. Their main result suggests that large voting bodies are more difficult to buy if only one party is trying to buy the election. This is intuitive. However, they argue that if multiple parties are trying to buy the election then it might be possible that buying larger voting bodies is actually easier. They evaluate potential countermeasures to vote buying and conclude that the secret ballot is an effective strategy. Several other articles look at other means to manipulate voting [33, 32, 20]. These include payments to individuals to show up to the polls (turnout buying), but not contracting on the actual vote. Morgan and Várdy [33] investigate a vote buying situation in which players can pay voters to abstain (negative turnout buying). They argue that such vote manipulation is never optimal under an open ballot, but can be effective under a secret ballot. Examining negative turnout buying is not appropriate in our context. An individual has two choices: provide intelligence to the government or stay silent. There is no equivalent to "staying home on election day." Finally, as we discuss in Chapter 3, we feel that an open ballot assumption is reasonable for an insurgency situation.

Along with the the theoretical voting models, there are numerous studies that investigate empirical results of vote buying. Heckelman [34] examines the effects of secrecy in the voting process. He compares election turnouts for each state before and after the adoption of a secret ballot. He shows that the secret ballot accounts for a 7 percent decrease in voter turnout. Callahan and McCargo [35] argue that money was the decisive factor in the July 1995 election in Thailand. Vicente [36] studies the effects of vote-buying in elections under a secret ballot. Results of his study on the general elections in newly oil-rich Sao Tome and Principe (Western Africa) suggest that vote-buying increases voter turnout and is a tactic often utilized by a challenger to counteract the incumbency advantage. This result concurs with the theoretical results of Groseclose and Snyder [6], where the player who moves second (who can be viewed as the incumbent) has a strategic advantage in the game.

In our study, we model coercion as a tool for the insurgents. In their study on violence during election campaigns, Collier and Vicente [37] separate the supporters of rival political parties into two main groups: hardcore and soft base supporters. They assume that coercion effectively eliminates soft base supporters. They are too intimidated to provide support. Hardcore supporters, however, will still continue to provide support even in the face of coercion. They show that weaker candidates will resort to violence (what they call terrorism). We find a similar result,

although our formulation of coercion is different than in [37]. They model coercion as a binary decision variable, whereas we allow the insurgents to vary the coercion level over a continuum. Furthermore, we do not distinguish between soft base and hardcore supporters, although it has potential to be incorporated in further extensions of our model. In our model, the coercion modifies the preferences of the entire society instead of just affecting the soft base supporters.

There are a wide range of social science studies that also investigate the existence and results of voter intimidation. Schedler [38] investigates the totalitarian regimes that hold elections and argues that if power and money determine electoral choices, then democratic freedom does not exist. Violence or the threat of it can keep voters from exercising free choice. Kriger [39] examines the general elections in Zimbabwe and describes methods used to coerce voters:

> Methods of coercion against voters are extended from brutal "disciplining murders" as examples of the fate awaiting those who failed to conform, to generalized threats of retribution or a continuance or resumption of the war; to psychological pressures like name-taking and claims to the possession of machines which would reveal how individuals had voted; and to the physical interdiction of attendance at meetings. The universal longing for peace, and the ambience of recent violence, made the threats of general retribution or a continuance of the war a potent weapon even in the hands of unarmed activists, since it was independent of the secrecy of the ballot.

2.4 Groseclose and Snyder Model

We base our initial model on the work of Groseclose and Snyder [6] (henceforth referred to as just GS) and thus much of our base model is similar to their model. GS examine a continuum of voters who have some initial preference for voting for each candidate (or opposite sides of a bill). The candidates move sequentially with the challenger making the initial "bribe" offer and the incumbent moving second with a counter-bribe after observing the bribe of the challenger. They assume that the challenger is sufficiently strong to achieve victory (we do not make this assumption) and they examine the type of bribe offered by the challenger, how much it will cost the challenger to win, and the resulting support the challenger receives when he wins. The novel result from this work is that the winner will receive more than a majority of votes, and hence the term "supermajority" in the title of their paper. When both sides can offer bribes to the population, the winning side needs to have a sufficient buffer of voters to ensure his opponent cannot make a winning counter-bribe.

We describe our model in Chapter 3, and the base model closely parallels the GS model. One

of the goals of this thesis is to examine the insurgency scenario with a new modeling approach and lay the groundwork for future research. Aside from context and focus there are two main differences in our base model and the GS model

1. We examine an arbitrary victory threshold, whereas GS assume victory occurs with a majority of the votes.

2. In GS each individual has the same weight. This can be viewed as clout, or power, or influence; essentially one person, one vote. In our model we introduce a weighting function over the population such that different individuals have different influence on the outcome. This allows us to model the social structure of tribal areas in a more realistic fashion.

We then extend our base model in ways that do not appear in the GS paper. We incorporate coercion as an active parameter in the model. We also analyze the discrete population which receives little attention in GS. As mentioned in Section 2.3, the discrete version that examines weighted tribes is a much more complicated problem than what GS and others [29, 30] have studied.

2.5 Counterinsurgency Theory

Insurgents' dependence on popular support has been investigated in numerous recent studies. Gaga [40] examines historical evidence that suggests that the loss of support can hasten the demise of a terrorist group. His results concur with Crenshaw [41] who argues "cooperation or acquiescence from terrorists' erstwhile supporters gives the state better control over the operational terrain through improved access and intelligence, leading to greater terrorist attrition and disrupting their operations and organization." In his study on support for terrorism, Paul [42] examines how terrorists generate and maintain support and argues that "many of the same factors that motivate terrorist recruitment also motivate broad popular support for terrorists." He suggests that before developing and implementing an overarching plan to undermine the insurgents' support base, the government should identify the type of group, the extent of each group's support needs, and how it obtains these needs. Metz and Millen [43] investigate the "bandwagon" effect and argue that large segments of a population throw their support behind the side they believe will win. Helmus et al. [44] investigate the relationship between popular support and benefits provided to society, and they argue that provision of social services by the

insurgents has been found to be very effective in generating positive opinions, endorsement, and support for the insurgents. Flanigan [45] further notes that areas where a terrorist or insurgent group is the only provider of such services contain the highest levels of support for such organizations.

In our game, we assume that the insurgents have an objective and will behave in a manner to achieve that objective. Thus, there is some assumption of rationality regarding the insurgents and we must ensure this is a legitimate assumption. Berebi [46] argues that "terrorists are not particularly poor, ignorant, mentally ill, or religious. Their most notable characteristic is normalcy." He shows that at the tactical level, the evidence tends to support rational decision making by terrorists. Ganor [47], suggests that "In general, terrorist organizations usually conduct rational considerations of costs and benefits, but they often attribute different weight to the values taken into account in their cost-benefit calculations, and occasionally, may even consider values that are different from those of the ones coping with terrorism, thus making a decision that appears irrational to an outside observer." This observation agrees with Simon [48].

Several recent works have attempted to model the role of the population in insurgencies. Kress and Szechtman [49] examine the role collateral damage plays in strengthening an insurgency. They observe that the likelihood of collateral damage makes it extremely difficult to completely defeat an insurgency using only military force. The authors more explicitly model the population as a central player in insurgency situations in [50]. Here the government and insurgents provide benefits to the population, and the insurgents can coerce the population. The insurgents' ability to effectively coerce the population depends upon the intelligence they receive from their supporters. This leads to tipping points in the population. We also consider coercion and an imperfect ability for the insurgents to target the coercion. However, the coercion effectiveness parameter in our model is static. Epstein and Gang [51] analyze the counterinsurgency situation in which there are multiple insurgent groups vying for control of the general population along with the government. They examine who benefits when the insurgent groups share one common cause. Berman et al. [52] model the insurgency as a three-way interaction between insurgents seeking political change through violence, a government trying to minimize violence through some combination of service provision and force, and individuals deciding whether or not to provide the government with intelligence.

2.6 Social Structure in Afghanistan

Even though we primarily focus on the insurgency situation in Afghanistan, our model is general enough to apply to different insurgency situations. Our model can handle a social structure that consists of a handful of homogeneous tribes or a fairly heterogeneous population. As we will see the weight function can have a significant impact on the results, so it is important to explicitly model the population structure and not just assume every individual or tribe has equal clout. Unfortunately studies about the general social structure in Afghanistan have conflicting conclusions. A 2009 US ARMY TRADOC report [53] argues that the tribal structure in Afghanistan is not as strong as many believe, and counterinsurgency strategies that worked in a highly tribal Iraq may not work in Afghanistan. Another report by Gant [54] claims that success in Afghanistan depends crucially on dealing with the tribes because they have significant control over the population. Giustozzi [55] takes a more moderate view and argues that importance of Afghan tribes varies by region of the country. This view agrees with Ahmad [56].

CHAPTER 3: Model Formulation

In this chapter, we describe the model. We first discuss the settings and assumptions in Section 3.1. In Section 3.2, we present the initial model, and in Sections 3.3 and 3.4, we formulate extensions to the model that account for coercion and discrete populations, respectively. Most of the technical details appear in the Appendix.

3.1 Settings and Assumptions

We examine an insurgency situation where the the insurgents are in control of a region and government forces seek to defeat them. An example of this situation can be found in certain areas of Afghanistan in 2011. While the nature of this conflict is quite complex, we analyze only one part of it: the battle for the hearts and minds of the population. The insurgents need support from the population in terms of resources, recruits, and safe haven [40, 42]. The government needs the population's support primarily in the form of intelligence to effectively target the insurgents. In this model, we assume that the government "wins" the battle for the population if it receives intelligence from at least some certain fraction of the population. Otherwise the insurgents win, and therefore the outcome is binary. For example, if the threshold is 40%, then the government will win if it receives intelligence from at least 40% of the population, otherwise the insurgents win. This is of course an oversimplification, but it allows us to gain initial insight into the problem. We discuss this and other shortcomings in Chapter 5.

To model this interaction between the government and insurgent, we adapt a vote-buying model by Groseclose and Snyder (GS) [6]. In GS two individuals are running for office (or two parties are on opposite sides of a bill) and they can pay voters for their vote. In our model the government and the insurgents pay (i.e., bribe) the population for support. We define support for the government as providing it with intelligence. As in GS we formulate a two player, two period, sequential game. The government moves first and chooses an amount to bribe each individual in the population. The insurgents observe this bribe and then make a counter-bribe. We assume this sequence partly for analytic tractability, but also because it gives the insurgents a strategic advantage. If the insurgents are entrenched in an area, they have no incentive to take action until after observing the bribe of the government. While we define the strategies of the government and insurgents as bribes and counterbribes, respectively, the actual transfers do not necessarily

need to take this crude form. The government or insurgents could be providing benefits such as food, medical services, security etc.

Each individual in the population has an initial net-preference for providing intelligence to the government. This preference level can be negative, in which case the individual prefers staying silent. In the model, if an individuals stays silent they are supporting the insurgents. An individual compares the bribe offered by the government with the counter-bribe offered by the insurgents and, in conjunction with his initial preference, accepts the bribe from the side who provides him with the most value (i.e., utility) and offers his support. If enough individuals support the government, then the government wins, otherwise the insurgents win.

The insurgents and government put a certain value on achieving victory in the population domain. There are several potential interpretations of this value. This could represent a budget constraint on the amount of bribes each side can offer. Or it could represent the amount each side is feasibly willing to spend to win. The battle for the population is just one component of the insurgency conflict, and so each side cannot afford to spend an arbitrarily large amount on gaining the population's support. We prefer the latter interpretation. Under this interpretation, the valuation of victory corresponds to a measure of strength for each side.

We assume the government moves first because the insurgents are currently entrenched in the region and the government must actively engage with the population to gather intelligence. After observing the tactics of the government, the insurgents will counter. However, the insurgents have no incentive to preemptively bribe the population before the government tangibly manifests itself as a threat.

As discussed in GS, the net-preference level of an individual does not reflect an individual's preference for who wins if one individual is a negligible fraction of the population. In this situation, one person's actions will not affect the outcome (i.e., the probability an individual is pivotal is essentially zero [6]). Thus, the net-preference level reflects an individual's utility from the act of providing intelligence to the government, after the cost of cooperation is discounted. This could reflect satisfaction from providing the location of an insurgent hideout to the government or perhaps guilt or shame from not providing assistance. The preference function can also capture fear of retaliation from the insurgents for those who cooperate with the government, although we more explicitly model coercion in Section 3.3. For a discrete population (see Section 3.4) the preference level can account for who an individual prefers to win because that individual can influence the outcome. A potential extension to this work would incorporate the

concept of conferring benefits on the population contingent on victory, which is similar to the campaign promise model of [31].

One important issue with traditional vote-buying is the credibility in the transaction. That is the voter can take a candidate's money, and if the voter cannot be effectively monitored then the voter can still vote for whoever she prefers, regardless of the payment. With the adoption of the secret ballot in many elections, classic vote-buying cannot be enforced [33, 32, 20]. However, we assume that the once an individual commits to a side by accepting their bribe the individual will follow through with the support. This is reasonable from the government's perspective: the government directly observes whether an individual provides intelligence and can make the bribe contingent on receiving the intelligence. The insurgents' ability to confirm that an individual is not providing intelligence to the government may not be as strong. However, we assume that the insurgents have intelligence that allows them to determine (perhaps imperfectly) who in the population reneges on the agreement and provides information to the government. We assume individuals do not want to face the consequences of taking the insurgents' bribe and then providing intelligence to the government.

We defer discussing the validity of the assumptions and other model shortcomings until in Chapter 5.

3.2 Base Model

In this section, we describe the base model and the solution to the two player game. The initial formulation modifies the GS model to fit within the insurgency context. The two modifications we make to the GS framework in this section are (1) having an arbitrary victory threshold (as opposed to majority), and (2) each individual has his own weight in which he can influence the outcome (in GS individuals have equal weight). We first present the model preliminaries and notation in Section 3.2.1. We next define the insurgent and government strategies in Sections 3.2.2–3.2.3. Finally we present the complete solution to the game in Section 3.2.4 and provide some analysis of the solution in Section 3.2.5.

3.2.1 Preliminaries

To achieve victory, the government requires a fraction L of the population to provide it with intelligence. We discuss the limitations of this assumption further in Chapter 5. We assume that there is a continuum of individuals and we parametrize individuals in the population by the parameter θ. Without loss of generality assume that $\theta \in [-\frac{1}{2}, \frac{1}{2}]$ (as in GS). An individual

θ has weight $w(\theta)$ and net-preference $v(\theta)$ for the government. As discussed in Section 3.1, this preference represents the utility an individual derives from providing the government with intelligence. If the preference value is negative, then the individual inherently prefers supporting the insurgents by not providing intelligence to the government. We assume $v(\theta)$ is in monetary units to be consistent with the monetary bribes provided by the government and insurgents. Hence utility is linear in money in this formulation. The total cost required to convince an individual to change his support is $|v(\theta)w(\theta)d\theta|$. That is, if an individual inherently prefers the insurgents such that the $v(\theta) < 0$, then the government would need to bribe this individual with an amount of at least $-v(\theta)w(\theta)d\theta$ to convince him to provide the government with intelligence (presuming the insurgents do not offer a counter-bribe).

As in GS, we sort the individuals in the population in nonincreasing order by their preference $v(\theta)$. Thus, $\theta = -\frac{1}{2}$ is the most pro-government individual, and $\theta = \frac{1}{2}$ correspond to the most pro-insurgent individual. We assume the weight function is defined by some continuous density $w(\theta)$ on $\theta \in [-\frac{1}{2}, \frac{1}{2}]$ with corresponding CDF $W(\theta)$. We will define $v(\theta)$ as a strictly decreasing continuous function, because the weight function can account for multiple individuals with nearly identical preferences. The individual defined by $\theta_N = v^{-1}(0)$ has neutral preferences (i.e., is indifferent between supporting the government and insurgents). All individuals with $\theta < \theta_L = W^{-1}(L)$ represent the fraction L of the population with the most pro-government preferences. Thus, if $\theta_L < \theta_N$, the government will receive enough support to win if the insurgents do not offer any counter-bribes. Otherwise the government needs to make an initial bribe to some individuals in order to win.

We model the interaction as a two stage sequential game. In the first stage, the government makes a bribe to the population. Where governments bribing strategy, $b_G(\cdot)$ is a function of θ. In the second stage, the insurgents observe the bribe of the government and offer a counter-bribe $b_I(\theta)$. If an individual θ accepts the government's bribe, he will provide intelligence to the government and receive utility $v(\theta) + b_G(\theta)$. Otherwise, he accepts the insurgents' bribe, does not provide intelligence to the government, and receives utility $b_I(\theta)$. Therefore, the government will only receive intelligence from an individual if $v(\theta) + b_G(\theta) > b_I(\theta)$. The outcome of the

game is thus

$$\text{government victory if } L \leq \int_{v(\theta)+b_G(\theta)>b_I(\theta)} w(\theta)d\theta \tag{3.1a}$$

$$\text{insurgent victory if } L > \int_{v(\theta)+b_G(\theta)>b_I(\theta)} w(\theta)d\theta \tag{3.1b}$$

The objective of the game for both the government and insurgents is to win at the lowest cost. However, the government values the victory at V_G and the insurgents value the victory at V_I. If the cost to achieve victory is greater than its valuation, the side prefers to concede rather than obtain a costly victory. As discussed in Section 3.1, the quantities V_G and V_I are a measure of the strengths of the government and insurgents, respectively.

We summarize the notation of the model in Table 3.1.

Parameter	Description
L	amount of support the government needs for victory
θ	parameter corresponding to an arbitrary individual in the population
$v(\theta)$	preference function of population for government
$w(\theta)$	weight function of population
$W(\theta)$	CDF corresponding to $w(\theta)$
θ_N	$= v^{-1}(0)$, parameter of individual that has neutral preferences
θ_L	$= W^{-1}(L)$, the Lth percentile of the population in terms of their preference for the government
$b_G(\theta)$	government bribe function
$b_I(\theta)$	insurgent counter-bribe function
V_G	the value the government places on victory (strength of government)
V_I	the value the insurgents places on victory (strength of insurgents)
S_G	support government would receive in absence of insurgent counter-bribe
m	government bribe parameter (see Sections 3.2.3)
$H(m)$	height of leveling bribe (see Section 3.2.3)
$T_G(m)$	cost to implement government bribe (see Section 3.2.3)
α	insurgents' level of coercion
p	situational awareness parameter related to coercion (see Section 3.3.2)

Table 3.1: Model Notation

Next, we define several terms to distinguish between the concepts of attitude and behavior. An individual's attitude is dictated by his religion, cultural background, tribal affiliation, etc., and we assume it is fixed throughout. In our model, an individual's attitude is either pro-insurgent or pro-government. The behavior of an individual is how they act, and in our model it is manifested by whether whether he supports the government or support the insurgents. That is providing (or withholding) intelligence determines behavior. The attitude and behavior of an individual need not coincide. See [50] for a further discussion of attitude and behavior. Below we define these terms to clearly denote how they will be used throughout this chapter.

Definition 1. ***pro-government****: An individual parameterized by θ is pro-government if $v(\theta) > 0$*

Definition 2. ***pro-insurgent****: An individual parameterized by θ is pro-insurgent if $v(\theta) < 0$*

Definition 3. ***government supporter****: An individual supports the government if they will provide the government with intelligence. That is if $v(\theta) + b_G(\theta) > b_I(\theta)$.*

Definition 4. ***insurgent supporter****: An individual supports the insurgents if they do not provide intelligence to the government. That is if $v(\theta) + b_G(\theta) \leq b_I(\theta)$.*

In the remainder of this section, we examine the equilibrium of this game and the optimal strategy of each side. As this section is primarily modifying the GS model into the insurgency context, some of the analysis borrows heavily from their work. Where appropriate, we will point out the similarities. As the insurgents move first, we first examine their best response to the government in Section 3.2.2. We next examine the government's optimal bribe in Section 3.2.3. This is the primary focus of the GS work, so this section most closely resembles their paper. We specify the complete equilibrium of the game in Section 3.2.4 and perform some additional analysis in Section 3.2.5

3.2.2 Insurgent Strategy

To solve this two player game, we first examine the insurgents, who move second. In the first period the government offers a bribe $b_G(\theta)$. If insurgents do not offer a counter-bribe, the government will have support S_G given by

$$S_G = \int_{v(\theta)+b_G(\theta)>0} w(\theta) d\theta. \tag{3.2}$$

We will initially ignore the constraint V_I and assume the insurgents will always make a winning counter-bribe. The insurgents will choose the lowest cost counter-bribe $b_I(\theta)$ such that the

government has less than L support from the population. That is, the insurgents solve the following optimization problem:

$$\min_{b_I(\theta)} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} b_I(\theta)w(\theta)d\theta \tag{3.3a}$$

$$L \geq \int_{v(\theta)+b_G(\theta)-b_I(\theta)>0} w(\theta)d\theta \tag{3.3b}$$

$$b_I(\theta) \geq 0 \tag{3.3c}$$

The optimal strategy for the insurgents is a greedy strategy, in which they bribe the weakest supporters of the government. The following proposition summarizes this strategy.

Proposition 1. *The lowest cost counter-bribe for the insurgents to win is to give a counter-bribe $b_I(\theta) = v(\theta) + b_G(\theta)$ to all individuals defined by the set $\{\theta \ : \ 0 < v(\theta) + b_G(\theta) < T\}$ and to some subset of the individuals defined by the set $\{\theta \ : \ v(\theta) + b_G(\theta) = T\}$ for some threshold value T.*

The proof of this appears in Section A.3 of the Appendix. Proposition 1 describes a simple strategy for the insurgents: sweep out the value T and counter-bribe everyone who requires less than T to stop providing intelligence to the government. Eventually, for large enough T, the insurgents will counter-bribe enough individuals such that the total fraction of the population providing intelligence to the government will drop below L. The optimal strategy defined by Proposition 1 is intuitive, but it does not hold in the discrete case (see Section 3.4).

Proposition 1 describes a lowest cost counter-bribe to achieve victory in terms of some threshold T, some set D, and a counter-bribe $b_I(\theta)$ for $\theta \in D$. For more details on the specific form of T and D see Section A.3 of the Appendix. The insurgents will execute this counter-bribe and win if the total cost of the counter-bribe is less than the insurgents' value of winning, V_I. Proposition 2 summarizes this.

Proposition 2. *Suppose the government makes a bribe $b_G(\theta)$ and a lowest cost counter-bribe by the insurgents involves bribing some set D of the population. Then the optimal response of*

the insurgents is to bid $b_I(\theta)$ given by

$$b_I(\theta) = 0 \qquad \textit{if } S_G \leq L \tag{3.4a}$$

$$b_I(\theta) = 0 \qquad \textit{if } S_G > L \textit{ and } \int_D (v(\theta) + b_G(\theta))w(\theta)d\theta \geq V_I \tag{3.4b}$$

$$b_I(\theta) = v(\theta) + b_G(\theta) \quad \textit{for } \theta \in D \quad \textit{if } S_G > L \textit{ and } \int_D (v(\theta) + b_G(\theta))w(\theta)d\theta < V_I \tag{3.4c}$$

The proof and the exact form of the set D appear in section A.3 of the Appendix. We more explicitly write out the insurgents' best response in Section 3.2.4 when we fully describe the equilibria of the game. Case (3.4a) of Proposition 2 corresponds to the situation where the insurgents win without needing to make a counter-bribe. Otherwise, the insurgents determine who they need to counter-bribe to win at the lowest cost according to Proposition 1. If this costs less than V_I, the insurgents execute a counter-bribe of $b_I(\theta) = v(\theta) + b_G(\theta)$ to this set of individuals and 0 to everyone else and win. This corresponds to case (3.4c). If the cost is more than V_I, victory is too costly so the insurgents do not offer a counter-bribe to anyone, and the government wins. This corresponds to case (3.4b).

3.2.3 Government Strategy

When analyzing the government's optimal bribe, there are several things to consider. First, the government must determine the lowest cost bribe it can offer and win. To do this it needs to decide who to bribe and how much to offer. After calculating the lowest cost bribe, the government next needs to determine if the bribe is worth the cost to implement it. In this section, we first describe the general form of the government's optimal bribe. We then specify the exact relationships necessary to produce the lowest cost winning bribe for the government.

General Government Strategy

As in section 3.2.2, we first assume that the constraint of V_G is not a factor and that the government will choose a bribing strategy $b_G^*(\theta)$ that results in a victory at the lowest cost. We first define a particular type of bribe by adopting the terminology in GS:

Definition 5. ***leveling bribe*** *(or strategy): A bribe $b_G(\theta)$ by the government is a leveling bribe if there is a constant a such that $v(\theta) + b_G(\theta) = a$ for all individuals the government bribes.*

Figure 3.1 illustrates a leveling bribe. Bribes of this type are relevant because all individuals for $\theta \in [\theta_0, m]$ in Figure 3.1 cost the insurgents the same amount to successfully counter-bribe.

This figure is similar to Figure 1 in GS.

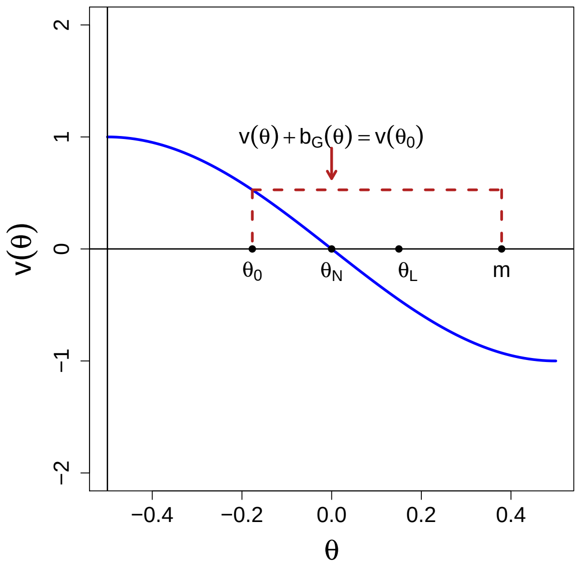


Figure 3.1: Leveling Strategy

The next proposition states thats the government's low-cost bribe can always be constructed as a leveling bribe.

Proposition 3. *An optimal bribe of the government can always be constructed as a leveling bribe. That is the bribe $b_G^*(\theta)$ satisfies $v(\theta) + b_G^*(\theta) = a$ for $\theta \in [v^{-1}(a), m]$ for some a and m. Furthermore $m \geq \theta_N$.*

The proof of this appears in Section A.4.1 of the Appendix. The intuition behind Proposition 3 is straightforward. The game is attacker/defender in nature and thus the insurgents will "attack the weak points" in the government's coalition under $b_G(\theta)$ (see Proposition 1). Therefore, the insurgents' strategic advantage in moving second naturally leads to a leveling strategy by the government in the first period. Moreover, Figure 3.1 illustrates that in many situations the individuals with moderate preferences may receive most of the payments, while those with extreme preferences receive nothing. The extremely pro-government individuals do not need any additional incentives to support the government, and the extremely pro-insurgent individuals are too expensive to include in the government coalition.

Note that Proposition 3 states that if the insurgents win they can always enact an optimal leveling bribe. However, in some situations there are nonleveling bribes that would also be optimal.

That is they cost the same to the government to implement and the lowest cost response by the insurgents costs the same as the counter-bribe to the leveling strategy. We will focus exclusively on the leveling bribe strategies of the government as it is optimal and intuitive. Although later in this section we will mention a situation where other optimal bribes exist.

We will now describe the four scenarios the government faces and what the form of the optimal bribe will be in each case. Recall we currently assume that V_G is large enough so that the government can win. By Proposition 3 the government will receive support from individuals for $\theta \in [-\frac{1}{2}, m]$ for some $m \geq \theta_N$. Let us redefine the level of support for the government in Equation (3.2) in terms of this parameter m

$$S_G(m) = \int_{-\frac{1}{2}}^{m} w(\theta)d\theta. \tag{3.5}$$

For the insurgents to win, they need to make a compelling counter-bribe to $max\{0, S_G(m) - L\}$ of the government supporters. To do this, the insurgents could bribe the individuals for $\theta \in [\theta_L, m]$ (see Corollary 2 in Section A.3 of the Appendix).

The following proposition describes the possible scenarios faced by the government when it wins. This proposition corresponds to Proposition 1 in GS (although it is framed within our context). The description of the 4 cases is notationally tedious. However, the cases are intuitive and we explain these four cases graphically after the proposition. It may be easier to refer to those figures rather than the notation in the proposition.

Proposition 4. *Define $b^*_G(\theta)$ to be a government's optimal bribe. One of the following four scenarios describes this optimal bribe.*

1. *If $\theta_L < \theta_N$ and $V_I \leq \int_{\theta_L}^{\theta_N} v(\theta)w(\theta)d\theta$, then $b^*_G(\theta) = 0$.*

2. *If $\theta_L < \theta_N$ and $\int_{\theta_L}^{\theta_N} v(\theta)w(\theta)d\theta \leq V_I \leq v(\theta_L)\int_{\theta_L}^{\theta_N} w(\theta)d\theta$, then there exists an optimal leveling bribe $b^*_G(\theta)$ such that $v(\theta) + b^*_G(\theta) = a \leq v(\theta_L)$ for $\theta \in [v^{-1}(a), \theta_N]$ and $b^*_G(\theta) = 0$ for $\theta \notin [v^{-1}(a), \theta_N]$. The value a satisfies $\int_{\theta_L}^{\theta_N} \max(a, v(\theta))w(\theta)d\theta = V_I$.*

3. *If $\theta_L < \theta_N$ and $V_I > v(\theta_L)\int_{\theta_L}^{\theta_N} w(\theta)d\theta$ then there is some leveling strategy $b^*_G(\theta)$ such that $b^*_G(\theta) = \frac{V_I}{S_G(m)-L} - v(\theta)$ on an interval where $b^*_G(\theta) > 0$ and $m \geq \theta_N$ and $\frac{V_I}{S_G(m)-L} \geq v(\theta_L)$.*

4. *If $\theta_L > \theta_N$ then there is some leveling strategy $b_G^*(\theta)$ such that $b_G^*(\theta) = \frac{V_I}{S_G(m)-L} - v(\theta)$ on an interval where $b_G^*(\theta) > 0$ and $m > \theta_L$.*

The proof of this appears in Section A.4.2 of the Appendix. Below, we explain intuitively the meaning behind the four cases with figures. These illustrations probably provide more insight then the proofs in the Appendix.

Case 1 of Proposition 4 corresponds to the case of the government having the inherent advantage and the insurgents being very weak (small V_I). Even if the government implements no initial bribe, the insurgents cannot mount an effective counter-bribe. By Proposition 1 the insurgents would counter-bribe the individuals in $[\theta_L, \theta_N]$, but by assumption the cost to do this is more than the insurgents are willing to spend. Figure 3.2 illustrates this case. We denote the area under the curve as A in Figure 3.2 and we give similar labels in the figures throughout this section. It is important to keep in mind that these areas are computed with respect to the density $w(\theta)$ so for example area $A = \int_{\theta_L}^{\theta_N} v(\theta)w(\theta)d\theta$ in Figure 3.2. This figure is similar to Figure 2 in GS.

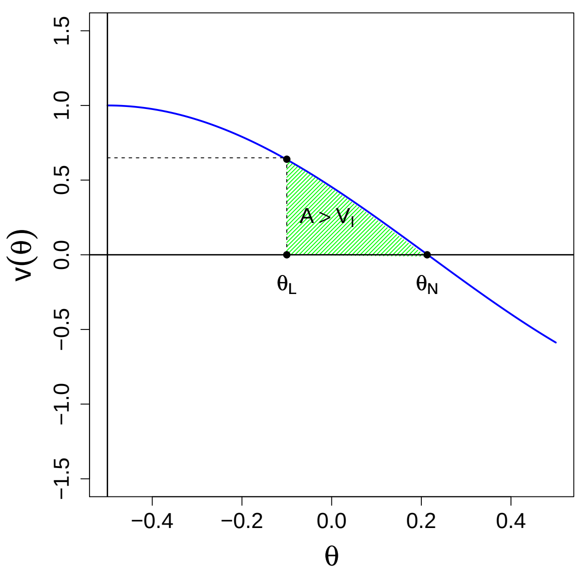


Figure 3.2: Case 1 of Proposition 4, area $A > V_I$. Total cost of government bribe equals 0

In Case 2 of Proposition 4, V_I is large enough such that the insurgents will overcome their initial preference disadvantage if the government does not make a bribe offer to the population. However, the government can achieve victory by bribing the individuals for $\theta \in [\theta_L, \theta_N]$, a modest amount. If the insurgents make a winning counter-bribe, by Proposition 1 the insurgents

would counter-bribe the individuals in $[\theta_L, \theta_N]$. Thus, the government needs to make the cost of this counter-bribe V_I to ensure victory. Figure 3.3 illustrates this case when the government executes a leveling bribe. This figure has similarities to Figure 3 in GS, but has some subtle differences.

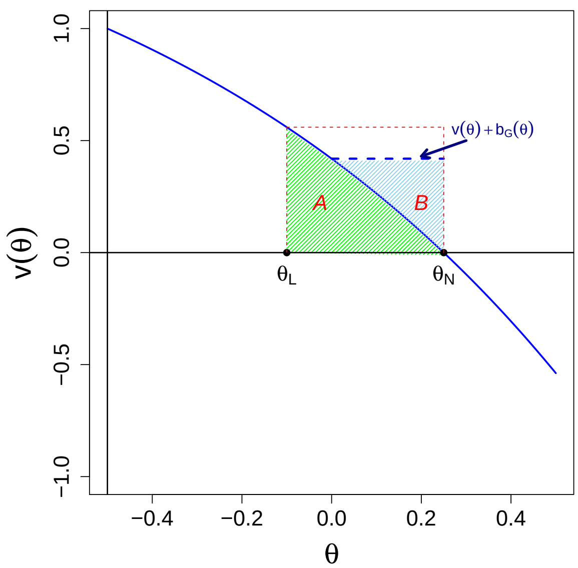


Figure 3.3: Case 2 of Proposition 4, area $A < V_I$. The government spends an amount B on bribes such that $A + B = V_I$

In Figure 3.3, the area A under the curve satisfies $A < V_I$ by assumption. Thus, the government needs to implement a bribe that adds an area B inside the red rectangle in Figure 3.3 such that $A + B = V_I$. The bribe in Figure 3.3 accomplishes this, however, any bribe that adds mass B to within the red rectangle will be an equivalent bribe. It would not be an optimal bribe for the government to implement a bribe $b_G(\theta)$ such that $v(\theta) + b_G(\theta) > v(\theta_L)$, because then by Proposition 1 the insurgents would counter-bribe individuals for $\theta < \theta_L$ and the government could reduce their bribe.

One of the observations in GS is that for case 2 of Proposition 4, the government only bribes pro-government individuals. This follows from the strategic nature of the interaction. The government cannot be content that it has an initial advantage; it needs to fortify its coalition to ensure the insurgents cannot make a successful counter-bribe. From a societal point of view, this is beneficial because the population gains benefits B (in Figure 3.3), even though the amount

of intelligence the government receives would be the same if bribes were not allowed by either side. As discussed in Section 3.1, the “bribes” could actually be societal benefits such as public goods, and therefore the conflict between government and insurgents produces additional benefits to society. In this case, the government would provide additional benefits even though the insurgents themselves would not. The possibility of the insurgents making a compelling counter-bribe is sufficient to cause the government to increase the benefits it provides to the society. This insight is consistent with Burgoon [57] who argues that terrorist groups can provide social welfare to marginalized groups in society in exchange for their support. When this occurs, the government must increase the provision of social benefits to these groups to decrease the incentives to support the terrorists.

In case 3 of Proposition 4, the government still has the initial preference advantage, but the insurgents are strong enough so that the government needs to bribe some of the pro-insurgent individuals to win. Case 4 of Proposition 4 is nearly identical to case 3 (at least in terms of the government’s response), but in this case the government initially is at a preference disadvantage. In both cases by Proposition 3 a leveling strategy is optimal. We illustrate case 4 in Figure 3.4. A figure illustrating case 3 would be identical to Figure 3.4, except the location of θ_L would be before θ_N and thus we omit it. The government needs to spend an amount D shown in figure Figure 3.4 to overcome the pro-insurgent preferences and then must spend an additional amount of $B + C$ to ensure the insurgents cannot offer a winning counter-bribe. Note that $C = V_I$, so the government must spend a considerable amount more than the insurgents are willing to spend to achieve victory. Thus, the government needs to be much stronger than the insurgents to win. From a counterinsurgency perspective, case 4 of Proposition 4 is the most realistic and interesting, because the insurgents have the initial advantage.

Lowest Cost Government Bribe

The first two cases of Proposition 4 uniquely determine the government’s optimal winning leveling bribe. However, cases 3 and 4 of Proposition 4 specify the leveling bribe in terms of a parameter m. In this section, we assume we are in case 3 or 4 of Proposition 4 and examine the optimal m^* that yields the lowest cost victory for the government. As before, we assume V_G is large enough that the government will win. First, let us define the height of the leveling bribe as a function of m:

$$H(m) = \frac{V_I}{S_G(m) - L}. \tag{3.6}$$

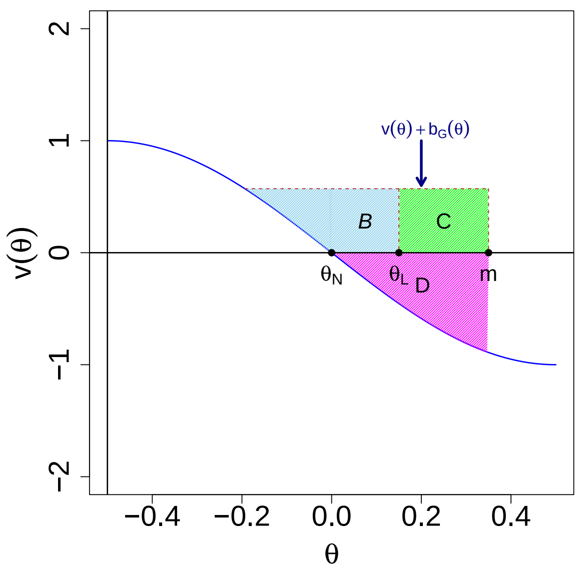


Figure 3.4: Case 4 of Proposition 4. The government spends an amount $B + C + D$ on bribes such that $C = V_I$

By construction, $H(m)$ corresponds to the optimal bribe by the following relationship $H(m) = v(\theta) + b_G^*(\theta)$ for all θ that receive a bribe from the government. By inspection, $H(m)$ is a decreasing function of m. If the government chooses a larger m then they will bribe more individuals, however the per capita bribe $H(m)$ will decrease. Thus, the total cost to execute the government's bribe may increase or decrease with m.

As in GS, we will distinguish between two types of bribes: one in which the government bribes all of the individuals who provide intelligence, and one in which the government only bribes a fraction of these individuals.

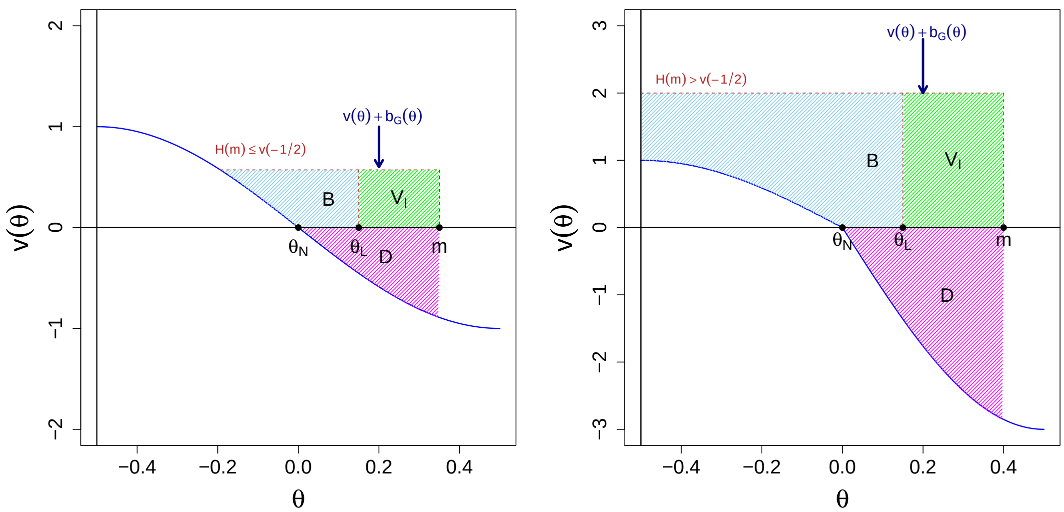
Definition 6. ***Flooded coalition***. *A coalition in which the government bribes all of its supporters. Formally the government supporting coalition is flooded if* $H(m) > v(-\frac{1}{2})$.

Definition 7. ***Nonflooded coalition***. *A coalition in which the government bribes only a fraction of its supporters. Formally the government supporting coalition is nonflooded if* $H(m) \leq v(-\frac{1}{2})$.

Definition 8. ***Universal coalition***. *All of the population supports the government. Formally* $m = \frac{1}{2}$ *and thus* $S_G(m) = 1$.

Figure 3.5 illustrates a flooded and nonflooded coalition. We have labeled the figures in the

same manner as Figure 3.4, although we have replaced C with V_I to emphasize that relationship. These figures are similar to Figures 4–5 in GS. We distinguish between these two types



(a) Nonflooded Coalition

(b) Flooded Coalition

Figure 3.5: Nonflooded and Flooded Coalitions. The total cost of the government's bribe = $B + D + V_I$

of winning coalitions because the mathematics to analyze the two situations is slightly different. In a nonflooded coalition, a set of the most pro-government individuals do not require a government bribe because their preferences so strongly favor the government that the cost to offer them a counter-bribe is too high for the insurgents even in the absence of a bribe from the government. In a flooded coalition, the value of V_I is large enough so that everyone in $[-\frac{1}{2}, m]$ requires a bribe from the government. We can now compute the total cost for the government to execute a bribe $b_G(\theta)$ for the flooded and nonflooded case:

$$T_G(m) = \begin{cases} \displaystyle\int_{-\frac{1}{2}}^{m} (H(m) - v(\theta))w(\theta)d\theta & \text{if } H(m) > v(-\frac{1}{2}) \quad (3.7a) \\ \displaystyle\int_{v^{-1}(H(m))}^{m} (H(m) - v(\theta))w(\theta)d\theta & \text{if } H(m) \le v(-\frac{1}{2}) \quad (3.7b) \end{cases}$$

The following proposition specifies the value m^* that minimizes the cost function $T_G(m)$ defined in (3.7a) and (3.7b). This corresponds to Proposition 2 in GS. As with Proposition 4, we provide a geometric interpretation of the relationships in Proposition 5 below

Proposition 5. *If the government wins under Case 3 or 4 of Proposition 4, the optimal value of m^* is unique and is one of the following*

1. *The government constructs a nonflooded nonuniversal coalition in which $H(m^*) \leq v(-\frac{1}{2})$, $m^* < \frac{1}{2}$ and m^* satisfies*

$$H(m^*)\left(L - W\left(v^{-1}\left(H(m^*)\right)\right)\right) = -v(m^*)(S_G(m^*) - L)$$

2. *The government constructs a flooded nonuniversal coalition in which $H(m^*) > v(-\frac{1}{2})$, $m^* < \frac{1}{2}$ and m^* satisfies*

$$H(m^*)L = -v(m^*)(S_G(m^*) - L)$$

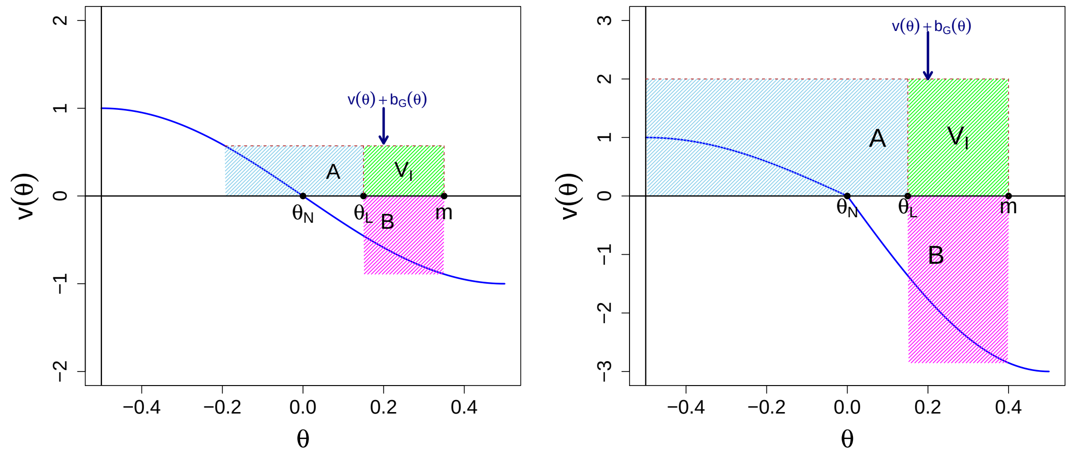
3. *The government constructs the universalistic coalition in which case $m^* = \frac{1}{2}$.*

The proof of this appears in Section A.4.3 of the Appendix. In the Appendix, we show there can be at most one solution to the root equations in case 1 and 2 of Proposition 5. Below we present figures that illustrate the conditions m^* must satisfy. If the winning coalition is not universal, then the bribe parameter m^* is chosen such that the area of rectangle A equals the area of rectangle B in Figure 3.6. As stated above the area is computed with respect to the density $w(\theta)$. Let us consider case 2 of Proposition 5, which corresponds to the flooded condition. The left-hand side of the equation in case 2 ($H(m^*)L$) is the area of A in Figure 3.6b and the right-hand side ($-v(m^*)(S_G(m^*) - L)$) is the area of B in Figure 3.6b. As m increases the area of A will decrease and the area of B will increase, so there is only one unique m^* where the two areas will coincide. Figure 3.6 is similar to Figures 6–7 in GS.

Once we compute the value of m^* as specified in Proposition 5, then we can substitute m^* into (3.7a) or (3.7b) to compute the cost for the government to win if they execute the lowest cost bribe defined by m^*. If $T_G(m^*) \leq V_G$ then the government should implement the bribe and achieve the victory. Otherwise the government should make no bribe and concede. This is summarized in Section 3.2.4.

3.2.4 Complete Equilibrium

Putting together the analyses in Sections 3.2.2 and 3.2.3, we can specify the equilibria of the game that will occur under different scenarios. In this section, we consider the value constraints



(a) Nonflooded Coalition (b) Flooded Coalition

Figure 3.6: Illustration of Proposition 5. At the optimum, A=B

V_I and V_G, which we ignored for most of Sections 3.2.2 and 3.2.3. The structure of these outcomes closely mirror those of Proposition 4

1. If $\theta_L < \theta_N$ and $V_I \leq \int_{\theta_L}^{\theta_N} v(\theta)w(\theta)d\theta$, then $b_G^*(\theta) = 0$, $b_I^*(\theta) = 0$ and the government wins.

2. If $\theta_L < \theta_N$ and $\int_{\theta_L}^{\theta_N} v(\theta)w(\theta)d\theta \leq V_I \leq v(\theta_L)\int_{\theta_L}^{\theta_N} w(\theta)d\theta$, then there exists an $a \leq v(\theta_L)$ such that $\int_{\theta_L}^{\theta_N} \max(a, v(\theta))w(\theta)d\theta = V_I$. One of two alternatives will occur

   (a) If $\int_{v^{-1}(a)}^{\theta_N}(a - v(\theta))w(\theta)d\theta \geq V_G$ then $b_G^*(\theta) = 0$, $b_I^*(\theta) = v(\theta)$ for $\theta \in [\theta_L, \theta_N]$. The insurgents win.

   (b) If $\int_{v^{-1}(a)}^{\theta_N}(a - v(\theta))w(\theta)d\theta < V_G$, the government makes the leveling bribe $b_G^*(\theta) = a - v(\theta)$ for $\theta \in [v^{-1}(a), \theta_N]$ and $b_G^*(\theta) = 0$ for $\theta \notin [v^{-1}(a), \theta_N]$ and $b_I^*(\theta) = 0$. The government wins.

3. If $\theta_L < \theta_N$ and $V_I > v(\theta_L)\int_{\theta_L}^{\theta_N} w(\theta)d\theta$ then there is some leveling strategy $b_G(\theta)$ such that $b_G(\theta) = H(m^*) - v(\theta)$ on an interval where $b_G(\theta) > 0$ and m^* is defined by the lowest cost bribe in Proposition 5. Define $\underline{\theta} = \inf\{\theta \in [-\frac{1}{2}, \frac{1}{2}] \ : \ b_G(\theta) > 0\}$. One of two outcomes will occur.

(a) If $\int_{\underline{\theta}}^{m^*}(H(m^*) - v(\theta))w(\theta)d\theta \geq V_G$ then $b_G^*(\theta) = 0$, $b_I^*(\theta) = v(\theta)$ for $\theta \in [\theta_L, \theta_N]$. The insurgents win.

(b) If $\int_{\underline{\theta}}^{m^*}(H(m^*) - v(\theta))w(\theta)d\theta < V_G$, the government makes the leveling bribe $b_G^*(\theta) = H(m^*) - v(\theta)$ for $\theta \in [\underline{\theta}, m^*]$ and $b_G^*(\theta) = 0$ for $\theta \notin [\underline{\theta}, m^*]$ and $b_I^*(\theta) = 0$. The government wins.

4. If $\theta_L > \theta_N$ then there is some leveling strategy $b_G(\theta)$ such that $b_G(\theta) = H(m^*) - v(\theta)$ on an interval where $b_G^*(\theta) > 0$ and m^* is is defined by the lowest cost bribe in Proposition 5. Again, we define $\underline{\theta} = \inf\{\theta \in [-\frac{1}{2}, \frac{1}{2} \ : \ b_G^*(\theta) > 0\}$. One of two outcomes will occur

(a) If $\int_{\underline{\theta}}^{m^*}(H(m^*) - v(\theta))w(\theta)d\theta \geq V_G$ then $b_G^*(\theta) = 0$, $b_I^*(\theta) = 0$. The insurgents win.

(b) If $\int_{\underline{\theta}}^{m^*}(H(m^*) - v(\theta))w(\theta)d\theta < V_G$, the government makes the leveling bribe $b_G^*(\theta) = H(m^*) - v(\theta)$ for $\theta \in [\underline{\theta}, m^*]$ and $b_G^*(\theta) = 0$ for $\theta \notin [\underline{\theta}, m^*]$ and $b_I^*(\theta) = 0$. The government wins.

In Chapter 4, we perform several numerical experiments to illustrate when each of these 7 outcome occurs, and how the results depend upon the various parameters.

3.2.5 Analysis

While we perform most of the analysis in Chapter 4, there are some analytic relationships we examine. Some of these results correspond to results in GS, whereas others are new contributions. In this section we assume that V_G is large enough so that the government will win. The next proposition describes the level of support that the government will receive if it wins.

Proposition 6. *If the government wins it must win with excess support. That is $S_G(m^*) > L$.*

The proof of this appears in Section A.4.4 of the Appendix. We only need to focus on case 4 of Proposition 4 when considering Proposition 6, because in cases 1–3 of Proposition 4 we have by assumption $\theta_L < \theta_N$, and thus $S_G(m^*) > L$. Within a game-theoretic framework the result of Proposition 6 is fairly intuitive. If $S_G(m^*) = L$, then the insurgents can make a winning counter-bribe at negligible cost. In GS, the analog of Proposition 6 is the major result. In many legislative scenarios, a measure passes with a large fraction of the votes. When examining the problem from a decision analytic framework, this seems suboptimal, because it would be less costly to win with the bare majority. GS's major contribution is that examining voting within

a vote buying context as a decision analytic process (as opposed to game theoretic) is flawed. This result is also useful from a counterinsurgency perspective. If the government (or coalition forces) need some base level of support or intelligence to effectively combat the insurgents, then they must realize that to accomplish this they will need to actually generate more than this base level. If the government only have the resources or tactics necessary to generate a level L of support, than the they will lose.

We next examine how the support level $S_G(m^*)$ varies with the other parameters of the model. The next Proposition describes those relationships

Proposition 7. *The level of support $S_G(m^*)$ the government receives for an optimal bribe has the following properties.*

1. *$S_G(m^*)$ is nondecreasing in V_I.*
2. *$S_G(m^*)$ is nondecreasing in L.*
3. *$S_G(m^*)$ is nondecreasing if the population's preferences shift from $v_1(\theta)$ to $v_2(\theta)$ and $|v_2(\theta)| \leq |v_1(\theta)|$.*
4. *$S_G(m^*)$ is nondecreasing if the population's weight shifts from $W_1(\theta)$ to $W_2(\theta)$, $W_2(\theta) \geq W_1(\theta)$ for all $\theta \in [-\frac{1}{2}, \frac{1}{2}]$ and the government's optimal bribe always produces a flooded coalition.*

The proof of this appears in Section A.4.5 of the Appendix. Case 1 of Proposition 7 is noted by GS and states that the stronger the insurgents are, the more support the government will receive if it wins. GS claims this is a counterintuitive result, but it should not be when viewing the interaction from a strategic standpoint. If the insurgents are strong, the only way the government can hope to win is if they can convince most of the population to support them, because this increases the cost of a winning insurgent counter-bribe. Cases 2-4 do not appear in GS. Case 2 of Proposition 7 is intuitive: if the government requires more support to achieve victory, then they amount that they actually receive will increase. Case 3 of Proposition 7 states that if the population's preferences are less extreme (i.e., pro-government individuals prefer the government less and pro-insurgent individual prefer the insurgents less), the level of support the government will generate increases. This follows for two reasons. If the pro-government individuals prefer the government less, then to implement the original bribe to those pro-government individuals would cost the government more. To decrease the cost, the government would lower the

level of the bribe, but in order to still achieve victory they would also have to increase m^* as a result (see equation 3.6). Furthermore, in this situation the bribes to convince pro-insurgent individuals to provide intelligence decreases, and thus the government can increase the number of those individuals that support the government. Finally, case 4 of Proposition 7 states that if the population's weight shifts so that a larger fraction of the population is pro-government, then the government will receive more support if it wins. This is intuitive, because the population's attitudes have shifted toward the government so it costs less to gain more supporters.

3.3 Coercion

In this section, we propose a modification to the initial model to incorporate insurgent coercion. We can view coercion as being an implicit component of the base model in Section 3.2 by modeling the preference function $v(\theta)$ as accounting for the possible coercion that individuals may receive for providing intelligence to the government. However, in that framework, coercion is a passive parameter. In this section, we model coercion as an active parameter that the insurgents control.

We first describe the formulation in Section 3.3.1 and then perform additional analysis for the situation when the government obtains a flooded coalition (see Definition 6 in Section 3.3.2).

3.3.1 General Case

The base model of base model in Section 3.2 is a two period sequential game where the government makes an initial bribe to the government and then the insurgents make a counter-bribe. In this section, we assume there is a three period game: the insurgents move first and choose some coercion level $\alpha \in [0, 1]$, then the government observes this coercion level and they make their bribe, followed by the insurgents' counter-bribe in the final period. We incorporate coercion into the model in this fashion partly because of convenience to fit within our framework, but we also feel it is a realistic assumption for the coercion level to be set at the start. We assume that the insurgents have the upper hand in the area and thus have made the population aware of the potential repercussions for helping the government. Also in terms of their interactions with the populations, the insurgents have established some atmosphere of violence, fear, and intimidation. Therefore, we assume the insurgents commit to a level of coercion at the beginning of the game. There are other ways to incorporate coercion into the model (e.g., having the insurgents choose the coercion level after the governments offer the initial bribe) and we discuss the shortcomings of our coercion model further in Chapter 5. We further assume that coercion is a no cost endeavor for the insurgents. This may be reasonable if, for example, the

insurgents can create a level of fear in the population via a handful of thugs patrolling a village with weapons. The actual cost may be low compared with paying individuals or providing benefits via the counter-bribes. Even if this is not a good assumption, it does provide a best-case scenario for the insurgents with respect to coercion. We could include a cost to coercion, but it would be an additional parameter to the model and we feel it does contribute much to the analysis. The purpose is examine the optimal level of coercion from the insurgents' point of view. Even in this relatively best case for the insurgents, the optimal level of coercion is often not the maximum amount the insurgents could inflict.

We incorporate coercion by changing the preference function $v(\theta)$ of the population. The most general modification would be to define the preference function $v(\theta, \alpha)$ to be an arbitrary function of the coercion parameter α. We assume coercion affects the population via the following relationships.

$$\frac{\partial v(\theta, \alpha)}{\partial \alpha} \leq 0 \quad \text{if } \theta < \theta_N \tag{3.8a}$$

$$\frac{\partial v(\theta, \alpha)}{\partial \alpha} \geq 0 \quad \text{if } \theta \geq \theta_N \tag{3.8b}$$

The conditions in (3.8) imply that as coercion increases the pro-government individuals prefer the government less and the pro-insurgent individuals prefer the insurgents less. As with other aspects of the model, part of the motivation of (3.8a)–(3.8b) is analytic convenience. However, we want to have a mechanism that accounts for coercion impacting the entire population, not necessarily just the pro-government individuals. Equation (3.8) implies that the intensity of preferences decreases for all individuals. The logic behind (3.8a) is that coercion effectively reduces the preferences of the pro-government individuals, because they fear violent reprisals from the insurgents. Thus the government may need to offer a larger bribe to these individuals to compensate them for the coercion they may receive as a result of supporting the government. Relationship (3.8b) states that the preference for the insurgents of pro-insurgent individuals also decreases as a result of the coercion. This may occur for several reasons. The insurgents presumably only want to coerce those providing intelligence to the government, but they may mistakenly coerce those who have pro-insurgent preferences. Furthermore, coercion creates an atmosphere of fear and violence and even if individuals do not like the government, their affinity for the insurgents may decrease as a result of living in a violent environment. To conclude, we want to incorporate coercion in such a way that it can negatively impact the pro-insurgent individuals (see Anbar Awakening [58]) and we feel that conditions (3.8a)–(3.8b) are a reasonable

first attempt to do that.

It is difficult to perform much analysis for an arbitrary function $v(\theta, \alpha)$ that satisfies (3.8), therefore to facilitate further analysis we examine a separable $v(\theta, \alpha)$ function:

$$v(\theta, \alpha) = \begin{cases} q_G(\alpha)v(\theta) & \text{if } \theta < \theta_N \quad (3.9a) \\ q_I(\alpha)v(\theta) & \text{if } \theta \geq \theta_N \quad (3.9b) \end{cases}$$

For some $0 \leq q_G(\alpha) \leq q_I(\alpha) \leq 1$ and $q_G'(\alpha), q_I'(\alpha) \leq 0$. The form in (3.9a)–(3.9b) implies that the relative impact of coercion on pro-government (or pro-insurgent) individuals is the same. We assume that $q_G(\alpha) \leq q_I(\alpha)$ so that coercion has a greater impact on the pro-government individuals than the pro-insurgent individuals. The insurgents should exert most of their coercion on pro-government individuals so the coercion should have a greater impact on them.

Once the insurgents choose the coercion parameter α, the preference function is determined by (3.9a)–(3.9b), and then the game proceeds as defined by the base model in Section 3.2. The equilibrium of the game will be the same as defined in Section 3.2.4, with $v(\theta)$ replaced by $v(\theta, \alpha)$ in (3.9a)–(3.9b). We will now examine the optimal level of coercion, α^*. Coercion may or may not affect the outcome of the game, and thus there may be multiple equilibria corresponding to many values of α^*. Rather than discussing those uninteresting cases, we will focus on the insurgents choosing α^* to maximize the the cost of the government's winning bribe. If this α^* increases the government's cost enough to change the equilibrium from government victory to insurgent victory (from case 3b or 4b to 3a or 4a in Section 3.2.4), there may be multiple levels of coercion that achieve this victory for the insurgents. Similarly, if the government's cost produced by α^* still allows government victory, than it doesn't matter what coercion level the insurgents choose (the government wins regardless). Thus the key component in the analysis is computing α^* that maximizes the government's initial bribe to achieve victory. That is what we will focus on here.

Our first two results do not examine α^*, but look at the specific cases when the government or insurgents will always lose because of coercion. Proposition 8 states that if the insurgents coercive powers are brutal enough and the government is weak, the government cannot overcome the fear in the population to achieve victory.

Proposition 8. *If $q_G(1) = 0$ and $\frac{V_I}{1-L} > V_G$ the government cannot win.*

The proof of this appears in Section A.5.1 of the Appendix. The condition $q_G(1) = 0$ in

Proposition 8 implies that the insurgents can break the will of the pro-government population with enough coercion (potentially at the risk of alienating the pro-insurgent population). The government can still win with a large enough bribe, but if $\frac{V_I}{1-L} > V_G$, they do not value the victory enough to offer a bribe. If the insurgents are strong and capable of very brutal coercion and the government is relatively weak, then the government cannot win. While Proposition 8 assumes the government is in a position of relative weakness, the next proposition assumes the insurgents are in the weak position, which corresponds to cases 1 and 2 of the equilibria defined in Section 3.2.4:

Proposition 9. *If $\theta_L < \theta_N$ and $V_I \leq v(\theta_L) \int_{\theta_L}^{\theta_N} w(\theta)d\theta$ and V_G is such that case 2(b) holds for the equilibria defined in Section 3.2.4 then one of the following holds.*

1. *If $V_I \leq q_G(1)v(\theta_L) \int_{\theta_L}^{\theta_N} w(\theta)d\theta$ the insurgents cannot win for any level of coercion.*
2. *If $V_I > q_G(1)v(\theta_L) \int_{\theta_L}^{\theta_N} w(\theta)d\theta$ the insurgents optimal level of coercion satisfies*

$$\alpha^* \geq \inf \left\{ \alpha \ : \ \frac{V_I}{v(\theta_L)(W(\theta_N) - L)} \geq q_G(\alpha) \right\}$$

The main condition of Proposition 9 corresponds to being in case 1 or 2 of the equilibria defined in Section 3.2.4. Case 1 of Proposition 9 implies that case 1 or 2(b) of the equilibria defined in Section 3.2.4 will still occur even if the insurgents maximize their coercion. This corresponds to a strong government and a weak insurgency. The insurgents are so weak that their coercion has a negligible impact on the population's preferences. One could argue that in this case the insurgents are so weak that they could be categorized as a nuisance more than an insurgency. Case 2 of Proposition 9 implies that the insurgents have enough coercive capabilities that they push the situation from case 1 or 2(b) to case 3 of the equilibria defined in Section 3.2.4. The optimal level α^* is nontrivial (i.e., greater than 0). Even though the insurgents are relatively weak, they effectively use coercion to put themselves on a more level footing with the government. This is similar to the insights in [37] where weak candidates in elections resort to violence. In the model of [37], the supporters of rival political parties split into two main categories: hardcore and soft base supporters. Coercion causes soft base supporters to withhold support for their party, while hardcore supporters will continue to provide support. While accounting for coercion in a different manner than in our model, they do derive a similar result that coercion is effective for weaker parties. Note that Proposition 9 does not imply the insurgents will win, but

it implies that the scenario will correspond to case 3 instead of case 2 in Section 3.2.4, which is more favorable to the insurgents.

We now focus on the specific value of α^* that will maximize the cost for the government to achieve victory. We first rewrite several of the terms from Section 3.2.3 in terms of α. We define $m(\alpha)$ to be the optimal m defined by Proposition 5, which now depends upon the coercion level α. The next proposition states how $m(\alpha)$ varies with α.

Proposition 10. *The government bribe parameter $m(\alpha)$ is a nondecreasing function of α*

This is just a direct application of case 3 of Proposition 7 in Section 3.2.5 and Lemma 4 of Section A.4.5 in the Appendix. Because the government's winning level of support $S_G(m) = W(m)$ is a nondecreaesing function, the following corollary follows immediately from Proposition 10

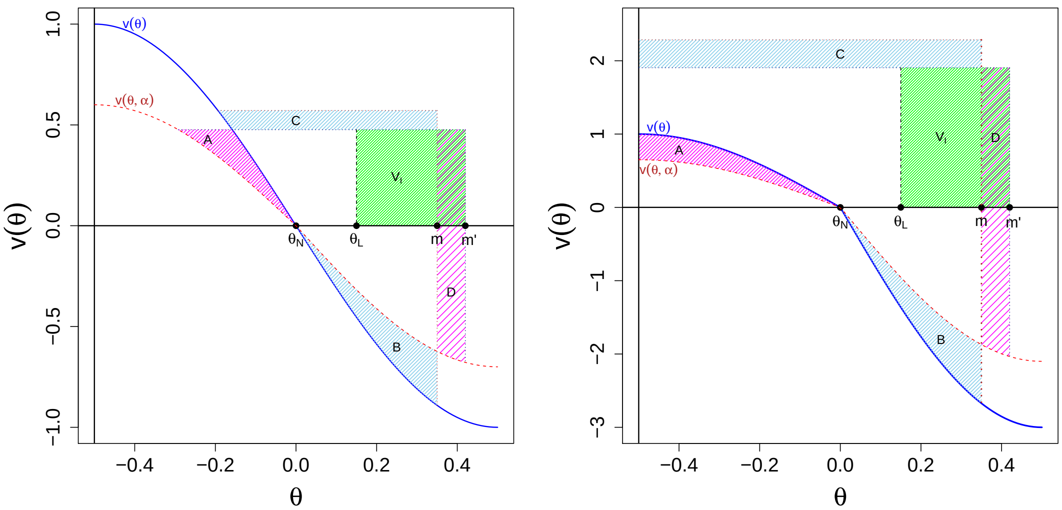
Corollary 1. *The government's winning support level is a nondecreasing function of α*

Corollary 1 implies that in the face of coercion the government will receive more support and intelligence (if they win of course). This follows for two reasons that can be seen in Figure 3.7. First, with increased coercion it costs the government more to bribe the pro-government individuals (area A), and thus they want to decrease the height of the leveling bribe and pay less (area C). But by definition of this height $H(m)$ in equation (3.6), the government must bribe more individuals if the height decreases (area D). Secondly, and more importantly, coercion decreases the cost for the government to bribe the pro-insurgent population (area B), thus the government can successfully bribe more of them when the insurgents increase coercion. This concept is consistent with the outcome of the Anbar Awakening [58]. The coercion by the insurgents was brutal, which led to eventually nearly full support for the government when the population turned on the insurgents.

Next we redefine the height of the leveling bribe in equation (3.6) in terms of α

$$H(\alpha) = \frac{V_I}{S_G(m(\alpha)) - L}. \tag{3.10}$$

Finally we write the cost to the government in terms of α. Recall the cost function has two cases



(a) Nonflooded Coaliton

(b) Flooded Coalition

Figure 3.7: Coercion's impact on the government's bribe

corresponding to the flooded and nonflooded coalitions

$$T_G(\alpha) = \begin{cases} H(\alpha)S_G(m(\alpha)) \\ \quad -q_G(\alpha)\displaystyle\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta - q_I(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta & \text{if } \frac{H(\alpha)}{q_G(\alpha)} > v(-\frac{1}{2}) \quad (3.11a) \\ H(\alpha)\left(S_G(m(\alpha)) - W\left(v^{-1}\left(\frac{H(\alpha)}{q_G(\alpha)}\right)\right)\right) \\ \quad -q_G(\alpha)\displaystyle\int_{v^{-1}\left(\frac{H(\alpha)}{q_G(\alpha)}\right)}^{\theta_N} v(\theta)w(\theta)d\theta - q_I(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta & \text{if } \frac{H(\alpha)}{q_G(\alpha)} \le v(-\frac{1}{2}) \quad (3.11b) \end{cases}$$

We can examine how the cost changes with α by referring to Figure 3.7. The cost will decrease an amount of $C + B$, but the cost of the bribe will increase an amount $A + D$, and it is not clear from the figure which of those effects will dominate. This suggests that coercion may or may not be an effective tool for the insurgents. We can more rigorously analyze the behavior of the optimal coercion α^* by examining the derivative $T'_G(\alpha)$:

$$T'_G(\alpha) = \begin{cases} -q'_G(\alpha)\displaystyle\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta - q'_I(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta & \text{if } \frac{H(\alpha)}{q_G(\alpha)} > v(-\frac{1}{2}) \quad (3.12a) \\ -q'_G(\alpha)\displaystyle\int_{v^{-1}\left(\frac{H(\alpha)}{q_G(\alpha)}\right)}^{\theta_N} v(\theta)w(\theta)d\theta - q'_I(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta & \text{if } \frac{H(\alpha)}{q_G(\alpha)} \le v(-\frac{1}{2}) \quad (3.12b) \end{cases}$$

Going from equations (3.11a)–(3.11b) to (3.12a)–(3.12b) is not trivial and the calculations

appear in Section A.5.2 in the Appendix. Without further assumptions, it is difficult to make concrete comments on the value of α^*. However, we can produce some qualitative insights by examining equations (3.12a)–(3.12b). The derivative $T'_G(\alpha)$ is made up of two terms: a positive term (the first integral) and a negative term (the second integral). Furthermore, these two terms have a nice interpretation. The first term represents the overall intensity of preferences from the pro-government individuals, and the second term represents the intensity of preferences from the pro-insurgent individuals that the government successfully bribes. The insurgents want the derivative positive to increase the cost for the government so they want the first term to dominate. This will occur when the pro-government population has strong preferences for the government and the pro-insurgent individuals have mild preferences for the insurgents. In this case, coercion will be effective. However, if the situation is reversed and the pro-insurgent population has strong preference intensity for the insurgents (so the second term in (3.12a)–(3.12b) dominates), than coercion can backfire. In this case, the coercion significantly alienates the pro-insurgent faction of the population and it may be easier for the government to bribe these individuals and obtain enough support from the population to win. Therefore, equations (3.12a)–(3.12b) illustrate the fragile nature of coercion and that arbitrarily increasing coercion may not be the right choice for the insurgents and may actually lead to victory by the government.

In the next subsection, we make more specific assumptions to examine the effects of coercion.

3.3.2 Flooded Coalition

In this section, we limit the analysis to the case of flooded coalitions to show further results. In this case we only need to consider equations (3.11a) and (3.12a). We make this assumption partly because those two equations are more analytically tractable than the nonflooded counterparts, but also because flooded coalitions correspond to a strong insurgency. The government will produce a flooded bribe for any coercion level α if $V_I > v\left(-\frac{1}{2}\right)(1-L)$. We next make a further assumption on the coercion functions $q_G(\alpha)$ and $q_I(\alpha)$:

$$q_I(\alpha) = p + (1-p)q_G(\alpha) \tag{3.13}$$

The parameter p can take on two potential interpretations. In one, p is a measure of the insurgents' situational awareness in distinguishing between pro-government or pro-insurgent individuals when executing coercion. The higher p, the less collateral damage the insurgents cause. Another interpretation of p is the willingness of the pro-insurgent population to live and ac-

cept an environment of fear and violence. As with many assumptions in the model, part of the motivation is analytic tractability, but we also wanted to examine the effect of the situational awareness (or population resilience) parameter p on the model. The next Proposition states the obvious relationship that insurgents prefer a larger value of p.

Proposition 11. *$T_G(\alpha)$ increases in p for any α.*

The proof of this appears in Section A.5.3 of the Appendix. We next substitute equation (3.13) into equation (3.12a) to examine the derivate of the government's cost.

$$T'_G(\alpha) = -q'_G(\alpha)\left(\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta + (1-p)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta\right) \tag{3.14}$$

We can now make specific statements about the optimal coercion level α^*. The next proposition states that there are only three options for α^*

Proposition 12. *The optimal α^* either corresponds to a unique interior point maximizer that is the root of equation (3.14) or it is an endpoint: $\alpha^* \in \{0, 1\}$.*

The proof of this appears in Section A.5.4 of the Appendix. The interesting result of Proposition 12 is that in some cases $\alpha^* < 1$. This is useful from a policy standpoint because the insurgents may push too far with their coercion alienating the population and this may allow the government to effectively bribe the population and win. For lower levels of coercion, it may have been too costly for the government to win. We conclude this subsection by analyzing how α^* varies with some of the other parameters of the model. The following proposition is similar to Proposition 7 in Section 3.2.5

Proposition 13. *The optimal level of coercion α^* has the following properties.*

1. *α^* is nonincreasing in V_I.*
2. *α^* is nonincreasing in L.*
3. *α^* is nondecreasing in p.*

The proof of this appears in Section A.5.5 of the Appendix. Case 1 of Proposition 13 states that the stronger the insurgents are the less they need coercion. This is consistent with Proposition

9, which stated coercion was an effective tool for weak insurgents. Coercion has negative consequences for the insurgents, and so if the insurgents are strong enough they need not resort to it (or can afford to execute lower amounts of it). Case 2 of Proposition 13 states that the more intelligence the government needs to win, the less the insurgents should coerce. In this case the government needs a larger coalition to achieve victory and so will need to bribe more of the pro-insurgent individuals to do this. Increasing coercion would only alienate these individuals and make it easier for the government to win. Finally, case 3 of Proposition 13 states an intuitive result. The higher the situational awareness of the insurgents, the more they insurgents should coerce. There are fewer negative consequences of coercion, so the insurgents should execute more of it.

3.4 Discrete Population

In some contexts, individuals belong to a social structure (e.g., tribe, clan), and essentially pledge to follow the will of the group's leader. Even if the population is large, if there are only a limited number of these tribes , assuming that the population is a continuum may be a poor assumption. In this section, we examine a discrete population where the population consists of a finite number of tribes. Tribe i has preference v_i and weight w_i. The weight may, for example, be a function of the number of individuals in the tribe. We can view these situations as the government and insurgents dealing directly with a limited number of tribal leaders. In this setting, we can analyze which tribes the government and insurgents should target for support.

We assume there are N tribes and that the cost to cause a tribe to switch from its natural preference is $|v_i w_i|$ and that the transaction is all or nothing. A fraction of a tribe cannot be successfully bribed. As stated above we can view the negotiations as taking place solely with the tribal leaders. There are several articles that examine the GS setting for discrete population, and those results and analyses are similar to the original GS work [29, 30]. However, all of these works assume that the tribes have the same weight, which is not the case in our setting. The discrete case with arbitrary weights is much more difficult to analyze than the continuous model, because the greedy approach used in the continuous model no longer holds. The analysis of the model introduced in Section 3.2 relies on Proposition 1: the insurgents will counter-bribe all of the lowest cost individuals until they have eroded the government's support sufficiently. This greedy approach only applies to a discrete population in the limit as the number of tribes increases and the weight of any one tribe goes to zero. For a large number of tribes, the greedy approach should be close to the optimal solution. We illustrate why the greedy approach does

not apply in the discrete case via a counterexample in the next subsection, when we discuss the insurgents' optimal strategy. Then in Section 3.4.2, we examine the government's optimal bribe. The analysis in this section is somewhat limited and there is opportunity for future work to derive more significant results.

3.4.1 Insurgent Strategy

Just as in the continuous case, we first examine the insurgents counter-bribe. We assume the government executed a bribe of $b_G(i)$ to tribe i, which would produce potential support:

$$S_G = \sum_{i=1}^{N} I(v_i + b_G(i) > 0)w_i \tag{3.15}$$

As in the continuous, case we initially ignore the constraint V_I and assume the insurgents will always make a winning counter-bribe. The insurgents will choose the lowest cost bribe $b_I(i)$, such that the government has less than L support from the population. That is, the insurgents solve the following optimization problem

$$\min_{b_I(i)} \sum_{i=1}^{N} b_I(i)w_i \tag{3.16a}$$

$$\sum_{i=1}^{N} I(v_i + b_G(i) > b_I(i))w_i \leq L \tag{3.16b}$$

$$b_I(i) \geq 0 \tag{3.16c}$$

We can reformulate the optimization problem in equation(3.16) as a knapsack type problem. To do this, we define x_i to be an indicator variable that is 1 if the tribe will not support the government. The insurgents choose the x_i such that the government has less than L support (i.e., $\sum_{i=1}^{N} w_i(1 - x_i) < L$), and the insurgents want to do this at the lowest cost possible. The

optimization problem is now:

$$\min_{x_i, b_I(i)} \sum_{i=1}^{N} b_I(i) w_i x_i \tag{3.17a}$$

$$\sum_{i=1}^{N} w_i(1 - x_i) \leq L \tag{3.17b}$$

$$b_I(i) \geq (v_i + b_G(i)) x_i \tag{3.17c}$$

$$b_I(i) \geq 0 \tag{3.17d}$$

$$x_i \in \{0, 1\} \tag{3.17e}$$

The optimization problem in (3.17) has a nonlinear objective function, however we know that for the optimal solution $b_I(i) = 0$ or $b_I(i) = v_i + b_G(i)$, and thus we can make some modifications so that the the optimization problem is a knapsack variant. Details of this modification and a dynamic programming algorithm to solve for the low cost insurgent counter-bribe are given in Section A.6.1 of the Appendix. The main take away is that these quantities can be solved in a straightforward manner.

Once the insurgents solve the optimization problem in (3.17), they know how much it costs to win. If this costs less then V_I, the optimal strategy for the insurgents is to execute the solution (3.17) as the winning counter-bribe. Otherwise, if the cost is larger than V_I the insurgents concede victory to the government without making a counter-bribe. Now that we have specified how the insurgents would calculate their lowest cost counter-bribe and whether they would execute this counter-bribe, we will give an example where the lowest cost counter-bribe for the insurgents does not consist of greedily bribing the lowest cost tribes. Thus, a discrete version of Proposition 1 does not exist. However, inspection of (3.17) does lead to an analog version of Proposition 1 if the weights are uniform (i.e., $w_i = \frac{1}{N}$). In that case, constraint 3.17b determines how many tribes to counter-bribe, and the objective function in (3.17a) and constraint(3.17c) specifies that the cheapest of those tribes should be counter-bribed an amount $v_i + b_G(i)$. This case was analyzed in [29, 30].

Let us now proceed with the counterexample. Assume that the government does not offer an initial bribe, $b_G(i) = 0$ for all i, and the tribes have the characteristics listed in Table 3.2. Furthermore let us set $L = 0.4$, and thus the insurgents must successfully counter-bribe $S_G - L = 0.19$ to win. By inspection, the insurgents should make a counter-bribe of $v_2 w_2 = 0.3$

Tribe Number	Weight w_i	Preference v_i
1	0.04	2.0
2	0.20	1.5
3	0.05	1.3
4	0.30	1.1
5	0.24	-1.0
6	0.17	-2.0

Table 3.2: Tribe characteristics

to convince tribe 2 to not support the government. This reduces the support to the government to 0.39, and thus the insurgents win. The optimal counter-bribe does not involve the tribe with the weakest preference for the government (tribe 4, $v_4 = 1.1$), nor does it involve the tribe with the smallest weight (tribe 1, $w_1 = 0.04$), nor does it involve the tribe that was the lowest cost to counter-bribe (tribe 3: $w_3 v_3 = 0.065$). As with the classic knapsack problem, a greedy approach will not produce the optimal bribe for the insurgents.

3.4.2 Government Strategy

We initially assume that V_G is large enough for the government to win. The government knows the insurgents determine their low cost counter-bribe according to the optimization problem in (3.17). If we define the value of that low cost counter-bribe as an explicit function of the government's bribe, $C_I(b_G(\cdot))$, then the government solves the following optimization problem.

$$\min_{b_G(i)} \sum_{i=1}^{N} b_G(i) w_i \tag{3.18a}$$

$$C_I(b_G(\cdot)) \geq V_I \tag{3.18b}$$

$$b_G(i) \geq 0 \tag{3.18c}$$

Unfortunately, the constraint in (3.18b) involves the output of an integer program (see Section A.6.1 in the Appendix) in (3.17) and thus is not trivial. In Section A.6.2 of the Appendix, we discuss how to formulate (3.18) as minimax optimization program. In the Appendix we conjecture that the transformed optimization problem can be solved using variant of Bender's decomposition [59]. In this case, the government's optimal bribe can be calculated numerically using most standard solvers if N is not too large. We illustrate the discrete model in Chapter

4 with numerical examples. However, deriving any analytic insight from the specific form of the optimization problem appears difficult. Thus, we conclude that the discrete population with nonuniform weights is a difficult problem.

CHAPTER 4:
Numerical Examples

In this chapter, we present the results of numerical experiments that examine several potential counterinsurgency contexts. The purpose of this chapter is to illustrate the results from Chapter 4 and emphasize the practical implications of those results. In Section 4.1, we describe the specific parameters we will use in this section. In Section 4.2, we examine the base case model corresponding to Section 3.2. We next illustrate the effect of coercion in Section 4.3. Finally, we analyze the discrete model from Section 3.4 via numerical examples in Section 4.4.

4.1 Parameters

The base model requires five input parameters to fully characterize the game between the government and the insurgents. We list these below, but a more full discussion of these parameters appears in Section 3.2.1

1. V_I: the strength of the insurgents
2. V_G: the strength of the government
3. $v(\theta)$: population preference function
4. $w(\theta)$: population weight function
5. L: support required for government victory.

Each combination of the input parameters corresponds to a specific context that will produce different strategies and outcomes in the game. For many of the figures in this chapter, we will vary V_I over some range. For the support required for government victory, we will primarily analyze one of the following four values

1. $L = 0.1$: the government needs a minimal amount of support to win
2. $L = 0.4$: the government needs to receive intelligence from slightly less than half the population to win

3. $L = 0.6$: the government needs to receive intelligence from slightly more than half the population to win

4. $L = 0.9$: the government needs support from nearly the entire population to win

We next list the preference functions and weight functions we will use in Sections 4.1.1 and 4.1.2, respectively. We conclude with a brief summary of the parameter values in Section 4.1.3.

4.1.1 Preference Function

We examine three different population preference functions. We define them as $v_1(\theta)$, $v_2(\theta)$, and $v_3(\theta)$ so we can use this notation later in the Chapter when referring to these various preference functions. The mathematical definitions are given below, followed by their graphical representation in Figure 4.1.

1. $v_1(\theta) = -2\theta$ (symmetric preference)

2. $v_2(\theta) = -0.543e^{\theta+1} + 1.8953$ (pro-government population)

3. $v_3(\theta) = -2.4663\log(\theta + 1) - 0.7095$ (pro-insurgent population)

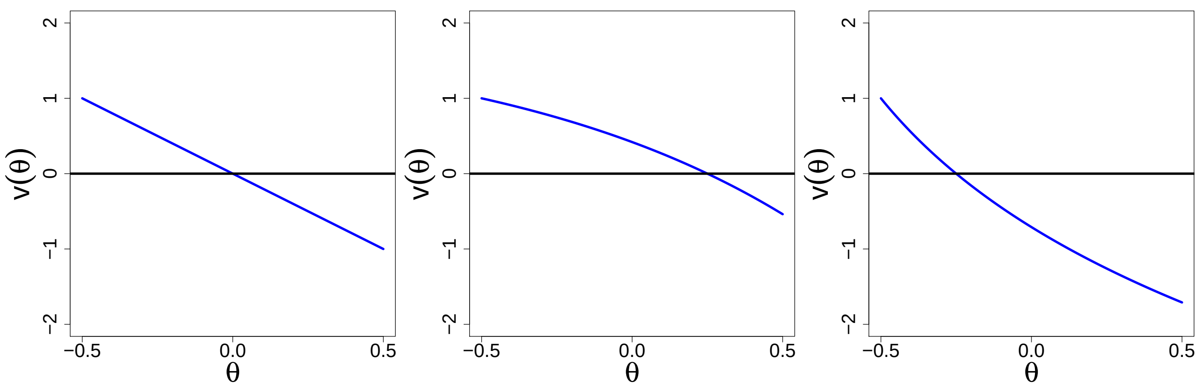
While the form of these three preference functions may appear odd, there is some logic to them. We define them such that $v_1\left(-\frac{1}{2}\right) = v_2\left(-\frac{1}{2}\right) = v_3\left(-\frac{1}{2}\right) = 1$ and that each one corresponds to either pro-government, neutral, or pro-insurgent preferences. Technically, the full preference of the population also requires the weighting function, but each preference function has a different value of θ_N. For $v_1(\theta)$ the neutral individual corresponds to $\theta_N = 0$, and the value is $\theta_N = 0.25$ and $\theta_N = -0.25$ for $v_2(\theta)$ and $v_3(\theta)$, respectively.

4.1.2 Weight Function

We examine 5 different weight functions and define them as $w_1(\theta)$, $w_2(\theta)$, $w_3(\theta)$, $w_4(\theta)$, and $w_5(\theta)$. We list the description of each weight function and present the densities in Figure 4.2.

1. $w_1(\theta)$: uniform

2. $w_2(\theta)$: triangular distribution with mode at $\theta = 0$. In this case, most of the population has fairly mild preferences for either the government or insurgents



(a) $v_1(\theta)$ (Symmetric) (b) $v_2(\theta)$ (Pro-government) (c) $v_3(\theta)$ (Pro-insurgent)

Figure 4.1: Population preference functions

3. $w_3(\theta)$: inverted triangular distribution with min at $\theta = 0$. In this case, the population has a bimodal distribution with two factions that have fairly extreme preferences.

4. $w_4(\theta)$: triangular distribution with mode at $\theta = -\frac{1}{2}$. In this case, the population is heavily weighted toward the individuals with larger preferences for the government.

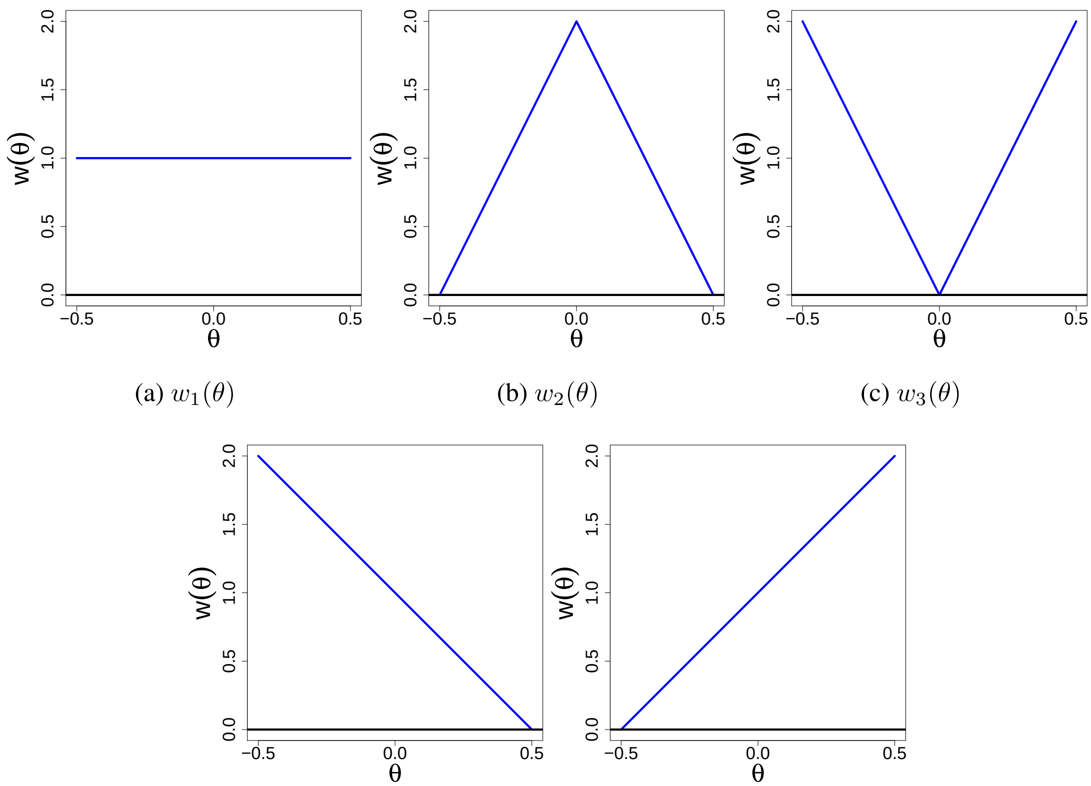
5. $w_5(\theta)$: triangular distribution with mode at $\theta = \frac{1}{2}$. In this case the population is heavily weighted toward the individuals with larger preferences for the insurgents.

While the weight function has to be combined with the preference function from Section 4.1.1 to determine the initial distribution of pro-government and pro-insurgent individuals, having different weighting functions allows us to examine different scenarios.

4.1.3 Summary

The purpose of this chapter is not to inundate the reader with 60 figures corresponding to the combinations of the preference function, the weight function, and the support required for government victory. We want to illustrate the model and its insights. Often times the results are fairly invariant to the specific combination and when they are not, we will highlight why that occurs. Having several different options for the parameters allows us to analyze several scenarios. Three examples appear below

1. Neutral scenario. Preference function = $v_1(\theta)$, weight function = $w_1(\theta)$. In this situation, 50 percent of the population is pro-government.



(a) $w_1(\theta)$ (b) $w_2(\theta)$ (c) $w_3(\theta)$

(d) $w_4(\theta)$ (e) $w_5(\theta)$

Figure 4.2: Population weight functions

2. Government advantage scenario. Preference function = $v_2(\theta)$, weight function = $w_4(\theta)$. In this situation, 93.75 percent of the population is pro-government.

3. Insurgent advantage. Preference function = $v_3(\theta)$, weight function = $w_5(\theta)$. In this situation, the 6.25 percent of the population is pro-government.

Obviously the government advantage scenario is not realistic in an insurgency context. The neutral scenario will probably not be realistic in many situations either. However, it will be interesting to compare how the relatively worst case (insurgent advantage scenario) differs from the relatively best case scenario (government advantage scenario).

4.2 Base Model

For various parameters, we would like to know who will win and if the government wins how much support it will receive. Determining who will win is equivalent to determining the min-

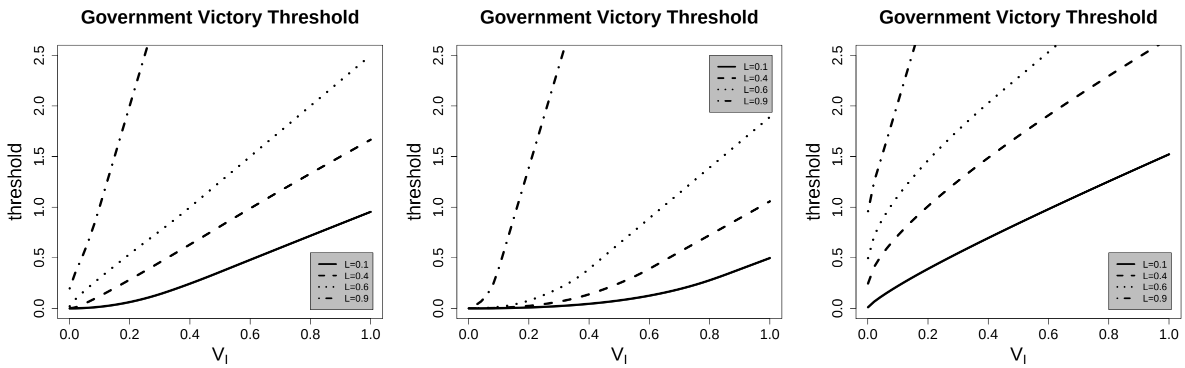
imum cost for the government to achieve victory. If V_G is less than this minimum cost, then the insurgents win, otherwise the government wins. We will study the two issues of victory (Section 4.2.1) and support (Section 4.2.2) in the following subsections and also illustrate the relationships from Section 3.2.5.

4.2.1 Government Victory Threshold

In Section 3.2.4, the outcome of the game is fully specified based on the model parameters. For parameters V_I, L, $v(\cdot)$, $w(\cdot)$, we can define $T_G(V_I, L, v(\cdot), w(\cdot))$ to be the minimum cost required for the government to achieve victory. This is the optimal value of $T_G(m)$ from equations (3.7a)–(3.7b). Computing $T_G(V_I, L, v(\cdot), w(\cdot))$ can be done via the analysis in Section 3.2.3 (specifically Proposition 5). In this section, we plot $T_G(V_I, L, v(\cdot), w(\cdot))$ as a function of V_I for fixed L, $v(\cdot)$, $w(\cdot)$. The quantity $T_G(V_I, L, v(\cdot), w(\cdot))$ forms a threshold: if $T_G(V_I, L, v(\cdot), w(\cdot)) < V_G$ the government wins, otherwise the insurgents win. By examining these plots, we see how much more difficult it will be for the government to win in certain scenarios. We will refer to $T_G(V_I, L, v(\cdot), w(\cdot))$ as the government victory threshold in this section.

We first examine how the government victory threshold varies with L in Figure 4.3. We only consider the neutral scenario (Figure 4.3a), the government advantage scenario (Figure 4.3b), and the insurgent advantage scenario (Figure 4.3c). These scenarios are described in Section 4.1.3. Obviously, the threshold increases with V_I: the stronger the insurgents are, the more the government will need to spend to successfully deter them from mounting a compelling counter-bribe. For large V_I, the curve describing the government victory threshold asymptotes to a line with slope $\frac{1}{1-L}$. This follows from equation (A.21a) in the Appendix and Lemma 4 of the Appendix. For large enough V_I, the government will eventually need a universal, flooded coalition of support to win (see Definitions 6 and 8). In that case, the lowest cost bribe of the government only depends upon V_I via a $\frac{V_I}{1-L}$ term. Proposition 7 states that the support for the government increases with L as shown in Figure 4.3, which is intuitive. The more supporters the government needs to procure the victory, the more it will need to spend. Furthermore, the difference in scenarios can be significant. The government victory threshold is nearly zero in the government advantage scenario (Figure 4.3b) for $L = 0.1$ for $V_I = 0.4$. While this case is beneficial for the government because they need limited support to achieve victory, the threshold is much larger (nearly 0.7) for the insurgent advantage scenario. In some cases, the threshold is 0 so the government can win without implementing a bribe (see Figure 4.3b for $L = 0.1$ and small V_I). This corresponds to case 1 of Section 3.2.4, where the insurgents are too weak to

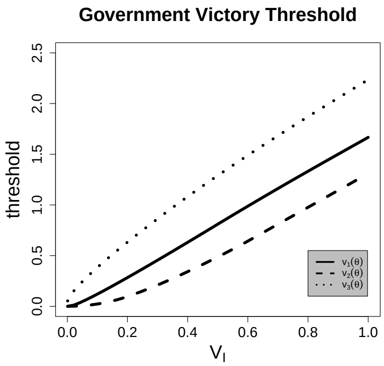
mount a compelling counter-bribe, even in the absence of a bribe by the government.



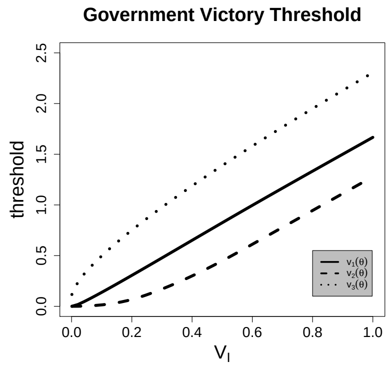
(a) Neutral: $v_1(\theta), w_1(\theta)$ (b) Gov. advantage: $v_2(\theta), w_4(\theta)$ (c) Insurgent advantage: $v_3(\theta), w_5(\theta)$

Figure 4.3: Government victory threshold as a function of V_I for several values of L

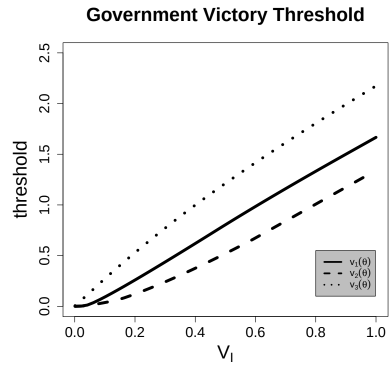
We next look at the case where we fix $L = 0.4$, and we plot the government victory threshold for the three preferences functions $v_1(\cdot)$, $v_2(\cdot)$, $v_3(\cdot)$. This appears in Figure 4.4. The combination of the preferences and weighting function determine the populations predisposition to the government and insurgents. The more the population prefers the government, the less government needs to pay for victory, which Figure 4.4 illustrates. Figures 4.4a–4.4c are similar, but there is a clear difference between Figures 4.4d and 4.4e because those represent the extreme weighting functions. This does illustrate the importance of considering the weighting function. Even if the majority of the population has pro-government preferences (i.e., we have $v_2(\cdot)$), if the influential people in the population are the individuals with stronger preferences for the insurgents (i.e., $w_5(\cdot)$), the government's threshold will be significantly higher. To see this, compare, for example, Figures 4.4a and 4.4e. Furtheremore, there is some subtle difference between Figure 4.4b and Figure 4.4c: the three curves are closer together in Figure 4.4c than in Figure 4.4b. If the the preference advantage favors the insurgents (e.g., $v_3(\cdot)$), then it is better for the government to have a relatively polarized population (e.g., $w_3(\cdot)$) so that a larger fraction of the population is initially pro-government. Thus, the threshold is lower for this case in Figure 4.4c, then Figure 4.4b. However, if the government holds a preference advantage (e.g., $v_2(\cdot)$), they do not want a polarized population (e.g., $w_3(\cdot)$) because that would increase the fraction of pro-insurgent individuals, which the government does not want given it has the initial position of power. Thus, the main take away from Figure 4.4 is that analysts should not arbitrarily assume the population is homogeneous without further justification.



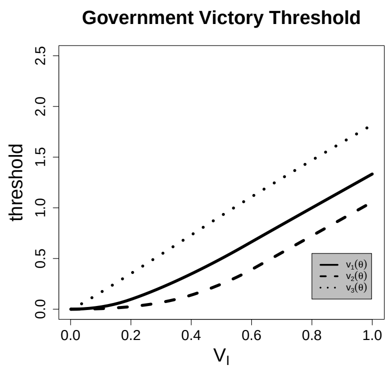
(a) L=0.4, $w_1(\theta)$



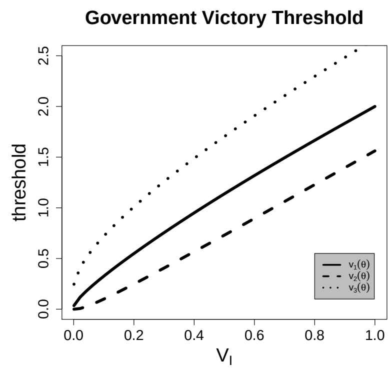
(b) L=0.4, $w_2(\theta)$



(c) L=0.4, $w_3(\theta)$



(d) L=0.4, $w_4(\theta)$



(e) L=0.4, $w_5(\theta)$

Figure 4.4: Government victory threshold as a function of V_I for several values of $v(\cdot)$

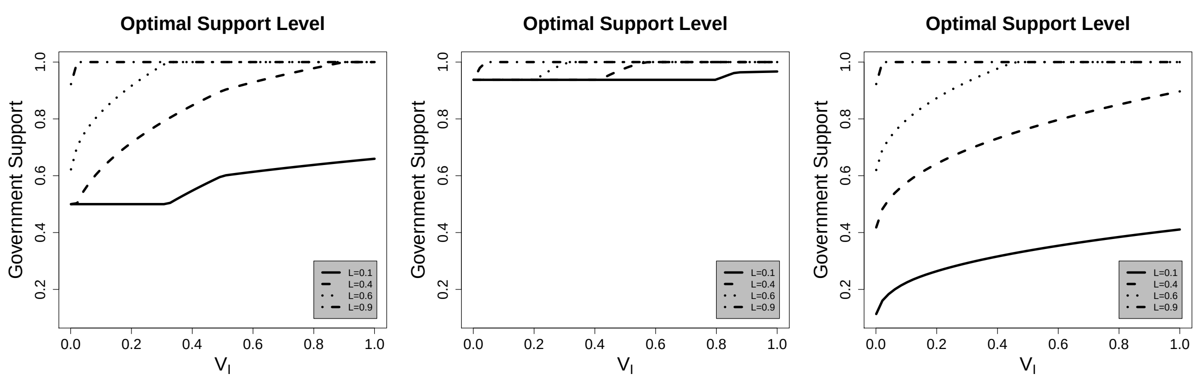
We could make a plot similar to Figure 4.4 that has a curve for each of the five weighting function. However, that figure is similar Figure 4.4 and provides the same insight, so we omit it.

4.2.2 Support Received by Victorious Government

In this section, we present plots showing how the support received by the government varies with the parameters. Here we assume the government strength is high enough so that it will win. That is, V_G is larger than the thresholds discussed in the previous subsection. In this section, we plot $S_G(m^*)$ where m^* is computed according to Proposition 5 and $S_G(m^*)$ is the level of support the government will receive if it executes a lowest cost winning bribe. We plot $S_G(m^*)$ as a function of V_I for fixed L, $v(\cdot)$, $w(\cdot)$.

As in Section 4.2.1, we first examine how the government victory threshold varies with L in

Figure 4.5. The level of support increases in V_I and in L in Figure 4.5, which is consistent with Proposition 7. Figure 4.5b is not very interesting, because it corresponds to the government advantage scenario where over 90% of the population has pro-government preferences. If the government wins, all pro-government individuals will support the government (see Proposition 4). Thus the government will always have very high support in this case. Much of the parameter space in Figure 4.5b corresponds to case 1 or 2 in Proposition 4, where the government supporters and the pro-government population are equivalent (i.e., $S_G(m^*) = W(\theta_N)$). Figures 4.5a and 4.5c are more interesting. The minimum possible support the government can receive in Figure 4.5a is 0.5 and in Figure 4.5c this value is 0.0625. Note that there is a kink in the $L = 0.1$ curve in Figure 4.5a around $V_I = 0.5$. This corresponds to the government's supporting coalition switching from a nonflooded coaltion to a flooded coalition (see Definitions 6 and 8).



(a) Neutral: $v_1(\theta), w_1(\theta)$ (b) Gov. advantage: $v_2(\theta), w_4(\theta)$ (c) Insurgent advantage: $v_3(\theta), w_5(\theta)$

Figure 4.5: Government support level vs. V_I

We omit showing the analog to Figure 4.4 because it provides the same insight as previous examples. For a fixed weight function, the government will receive more support for preference function $v_2(\cdot)$ than for $v_1(\cdot)$, and more support for preference function $v_1(\cdot)$ than for $v_3(\cdot)$. Also, as we discussed in the previous subsection, the weighting function has a significant impact on the results.

4.3 Coercion

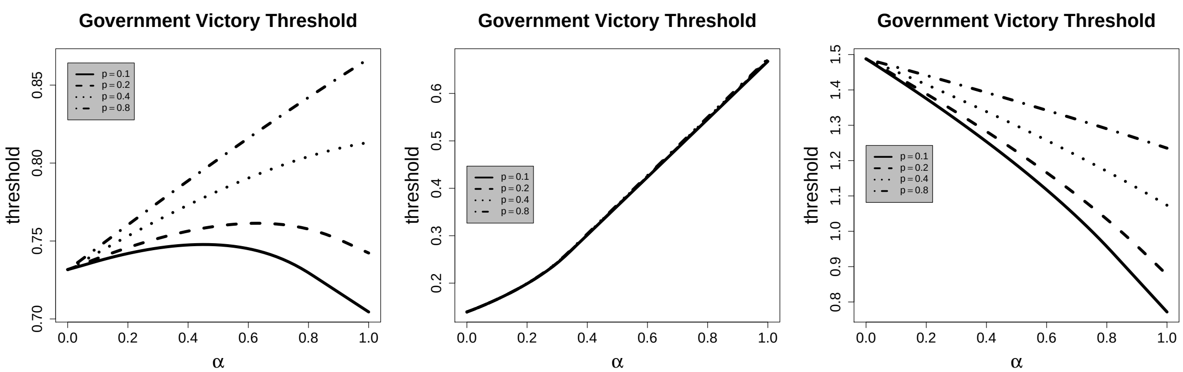
In this section, we examine the affect of coercion on the government's victory threshold and the support level conditioned on government victory. To determine this, we must compute the insurgents' optimal level of coercion. In this section, we will focus on the flooded coalition

case from Section 3.3.2. We use the same scenarios and parameters from Section 4.2, but must define two additional parameters from Section 3.3.2: the coercion effectiveness function $q_G(\alpha)$ and the situational awareness parameter p. We consider the simple function $q_G(\alpha) = 1 - \alpha$ and we vary p from 0.1 to 0.9. We first examine how coercion affects the government's victory threshold in Section 4.3.1 and then look at how the government's support varies with coercion in Section 4.3.2

4.3.1 Coercion's Impact on the Government Victory Threshold

Coercion effectively reduces the preferences of the pro-government individuals because they fear violent reprisals from the insurgents. Thus the government will need to offer a larger bribe to these individuals to achieve the same level of "defense" to prevent the insurgents from offering a compelling counter-bribe. As discussed in Section 3.3, coercion can backfire on the insurgents by reducing the pro-insurgent population's preferences. This makes it less costly for the government to entice the pro-insurgent individuals to provide them with intelligence. These conflicting effects of coercion are illustrated in Figure 4.6. For a given α, the government victory threshold is given by $T_G(\alpha)$ from equations (3.11a)–(3.11b). In Figure 4.6, we plot this threshold against α for $L = 0.4$ and $V_I = 0.4$. Each figure has curves corresponding to a different level of the situational awareness parameter p. This figure illustrates several important concepts. First, coercion is not universally good or bad for the insurgents. In the government advantage case in Figure 4.6b, the insurgents want to ramp up coercion because it has a significant impact. This is consistent with the discussion below Proposition 9, about coercion being a powerful tool for weak insurgents. However, if the insurgents are in the position of power (Figure 4.6c), coercion only serves to alienate the pro-insurgent population and make victory more likely for the government. Finally there may be some "sweet spot" level of coercion such as in Figure 4.6a, where coercion is effective for the insurgents up to a point. However, after this point the insurgents push too far and coercion backfires. This is consistent with the Anbar awakening where people speculate that the population finally snapped at the brutal insurgents and decided to help the coalition forces defeat the insurgents [58] Figure 4.6 also illustrates the main finding of Proposition 11, which states the victory threshold increases with the situational awareness parameter p. As the insurgents increase coercion the parameter p can have a significant impact on the threshold. The purpose of Figure 4.6 is to illustrate the three potential relationships between the threshold and coercion, and thus the y-axis of the three subfigures are not the same. Examination of the y-axis values reveals that while coercion can make a significant impact in the value of the threshold, the population's preferences and

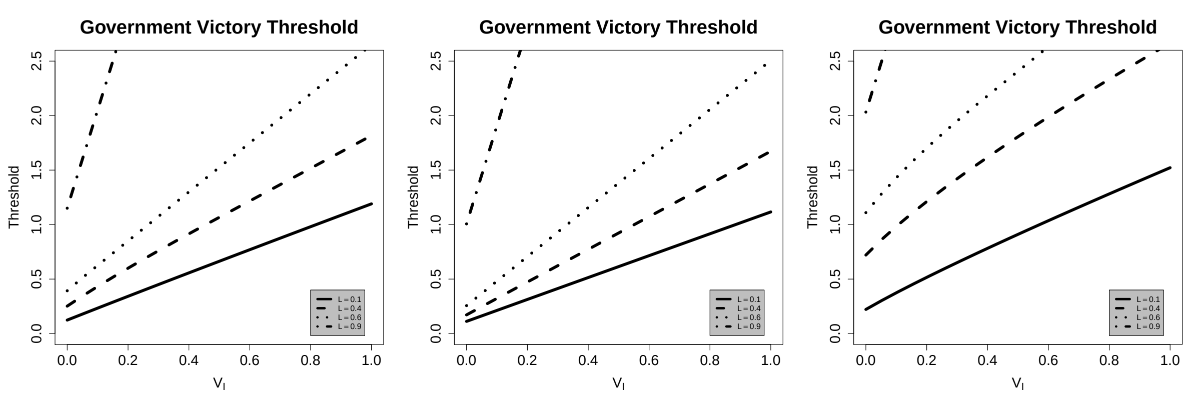
weights plays a larger role. Thus if the government can take actions to shift the preferences (or more unlikely weights), they should do that rather than spend those resources on reducing the coercing effectiveness of the insurgents.



(a) preference: $v_3(\theta)$, weight: $w_4(\theta)$ (b) Gov. advantage: $v_3(\theta), w_5(\theta)$ (c) Insurgent advantage, $v_3(\theta), w_5(\theta)$

Figure 4.6: Government victory threshold vs. coercion. $V_I = 0.4$ and $L = 0.4$

Next we plot the companion figure to Figure 4.3. That is, we plot the government victory threshold as a function of V_I. However, in Figure 4.7 we assume that the insurgents execute the optimal amount of coercion. That is, they choose the α^* to maximize the threshold (e.g., see Figure 4.6). As suggested by Figure 4.6, we see the potentially significant impact of coercion in Figure 4.7. Comparison with Figure 4.3 reveals the threshold increases dramatically in Figures 4.7a and 4.7b.

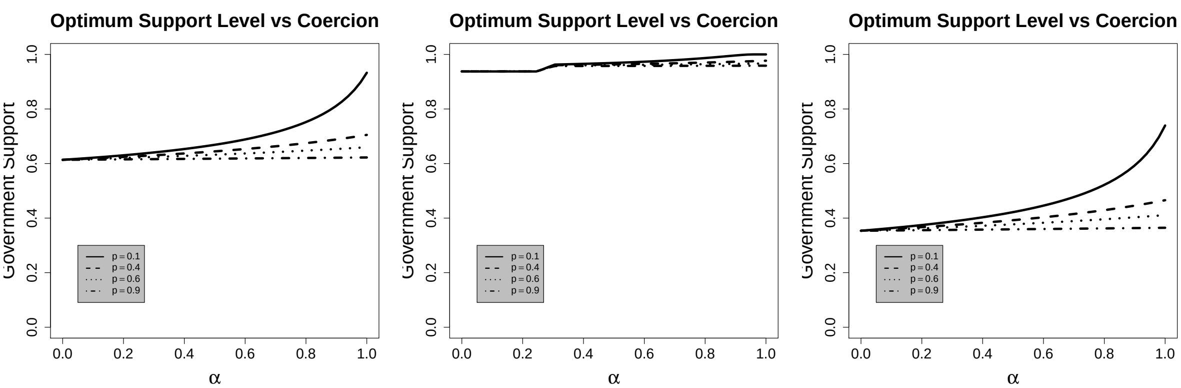


(a) Neutral: $v_1(\theta), w_1(\theta)$ (b) Gov. advantage: $v_2(\theta), w_4(\theta)$ (c) Insurgent advantage: $v_3(\theta), w_5(\theta)$

Figure 4.7: The government victory threshold as a function of V_I for several values of L, assuming the insurgents execute the optimal amount of coercion

4.3.2 Coercion's Impact on Government Support

In in this section, we examine how the government's level of support varies with coercion. In Figure 4.8 we set $V_I = 0.6$, $L = 0.1$ and choose several values of p. We choose different values for V_I and L than those used to create figure 4.6 so that we can better see the effects of coercion. We plot $S_G(m(\alpha))$ against α to see what impact coercion has on the level of support. While there is not a monotonic relationship between α and $T_G(\alpha))$ (see figure 4.6), there is an increasing relationship between α and $S_G(m(\alpha))$ in Figure 4.8. We proved this in Corollary 1. Coercion has minimal impact on the support level in the government advantage example in Figure 4.8b because the support level must be at least 0.9375. However, this is conditional on the insurgents winning after the coercion is executed. As discussed in Section 4.3.1, coercion has a significant impact on the government's ability to win in the government advantage scenario. We see the same importance of the situational awareness parameter p in Figure 4.8 as we did in Figure 4.6. Furthermore the support level decreases in the situational awareness parameter p, which follows from case 3 of Proposition 7.



(a) Neutral: $v_1(\theta), w_1(\theta)$ (b) Gov. advantage: $v_2(\theta), w_4(\theta)$ (c) Insurgent advantage: $v_3(\theta), w_5(\theta)$

Figure 4.8: Government support level vs. coercion. $V_I = 0.6$ and $L = 0.1$

The analog of Figure 4.5 for the optimal level of coercion provides similar insight and so we will not present that here.

4.4 Discrete Population

In this section, we examine a few scenarios for a discrete population. For consistency, we use the same preference and weight functions from Section 4.1.1–4.1.2, but discretize the population into 8 tribes. Tribe i is associated with $\theta \in [-\frac{1}{2} + 0.125(i-1), -\frac{1}{2} + 0.125i]$ for $1 \leq i \leq 8$. We define the preference v_i of tribe i to be $v(\theta_i)$ for θ_i corresponding to the right endpoint of the

corresponding region (i.e., $\theta_i = -\frac{1}{2} + 0.125i$). We define the weight w_i of tribe i to be the mass in the region with respect to the density $w(\theta)$. We list the parameters for discrete versions of the neutral scenario, the government advantage scenario, and the insurgent advantage scenario in tables 4.1–4.3.

Tribe Number	Weight $w_1(\theta)$	Preference $v_1(\theta)$
1	0.125	0.750
2	0.125	0.500
3	0.125	0.250
4	0.125	0.000
5	0.125	-0.250
6	0.125	-0.500
7	0.125	-0.750
8	0.125	-1.000

Table 4.1: Tribe characteristics (Neutral Scenario)

Tribe Number	Weight $w_4(\theta)$	Preference $v_2(\theta)$
1	0.234	0.881
2	0.203	0.746
3	0.172	0.593
4	0.141	0.419
5	0.109	0.223
6	0.078	0.000
7	0.047	-0.252
8	0.016	-0.538

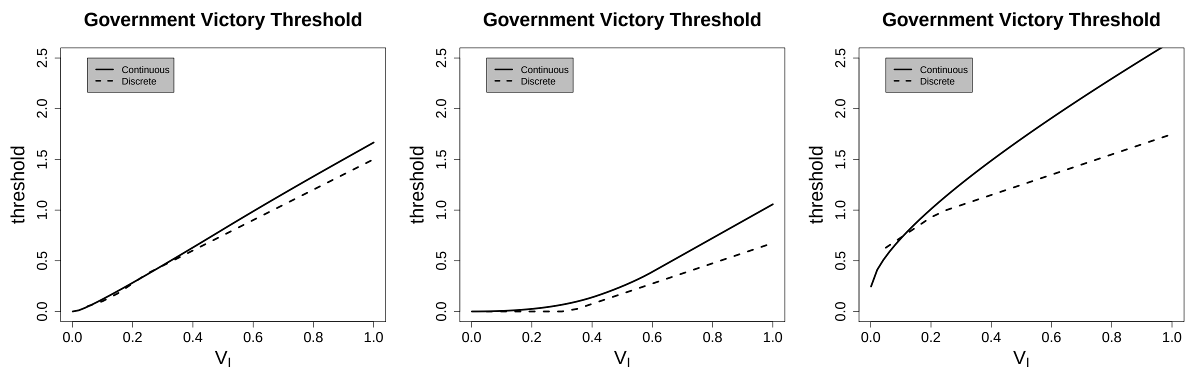
Table 4.2: Tribe characteristics (Government Advantage Scenario)

We first plot the victory threshold vs. the insurgents' strength V_I in Figure 4.9 for $L = 0.4$ for the discrete population and the continuous population. The curve corresponding to the continuous population first appeared in Figure 4.3. To solve for the government's optimal bribe for the discrete population, and the corresponding victory threshold, we use the algorithm described in Section A.6.2 of the Appendix.

The continuous and discrete scenarios are essentially equivalent in the neutral scenario in figure 4.9a. This agrees with the work of [29, 30] that found the continuous case and discrete case produce similar results for a discrete population with uniform weights. However, figures

Tribe Number	Weight $w_5(\theta)$	Preference $v_3(\theta)$
1	0.016	0.450
2	0.047	0.000
3	0.078	-0.380
4	0.109	-0.710
5	0.141	-1.000
6	0.172	-1.260
7	0.203	-1.495
8	0.234	-1.710

Table 4.3: Tribe characteristics (Insurgent Advantage Scenario)

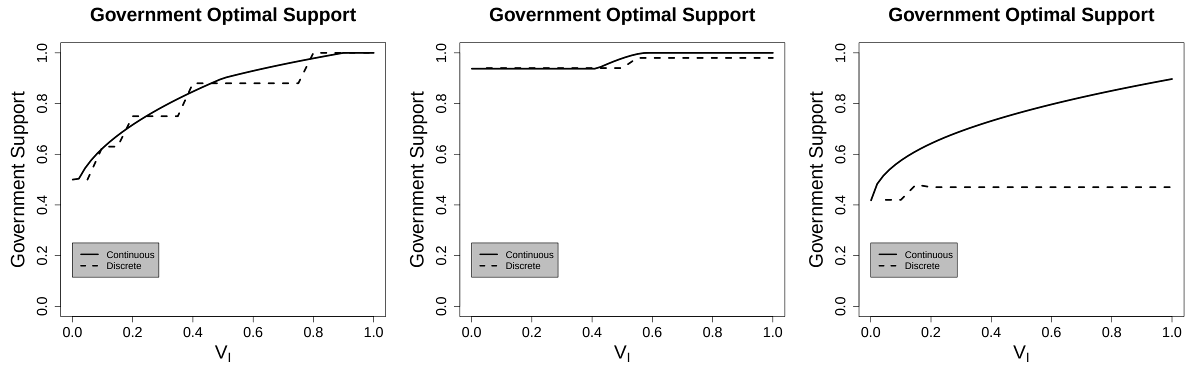


(a) Neutral: $v_1(\theta), w_1(\theta)$ (b) Gov. advantage: $v_2(\theta), w_4(\theta)$ (c) Insurgent advantage: $v_3(\theta), w_5(\theta)$

Figure 4.9: Government victory threshold: discrete vs. continuous population, $L = 0.4$.

4.9b–4.9c illustrate that nonuniform weights can have a significant impact on the results. This occurs because the weight function plays a more important role in the discrete case since the government and insurgents cannot gain the support of a fraction of the tribe, as they can in the continuous case. We examined similar plots to figure 4.9 for different values of L with various number of tribes. As expected from Proposition 7 and figure 4.3, the government victory threshold increases with L for the discrete population. A further expected result is that the difference between the continuous and the discrete case is more pronounced for a smaller number of tribes.

We next compare the level of support the government receives if it wins in figure 4.10. As with figure 4.5, the level of support increases in L for the discrete population. However, the support is not nondecreasing in V_I as it is in the continuous case (see Proposition 7). This can



(a) Neutral: $v_1(\theta), w_1(\theta)$ (b) Gov. advantage: $v_2(\theta), w_4(\theta)$ (c) Insurgent advantage: $v_3(\theta), w_5(\theta)$

Figure 4.10: Government support level: discrete vs. continuous population, $L = 0.4$.

be seen around $\theta = 0.15$ in figure 4.10c. We defer further discussion of this issue until we present figure 4.11 below. For most values of θ figure 4.10, the level of support is less for the discrete population than the continuous population. In these situations, it is either to costly for the government to gain more support, or the government does not need to create as large of a buffer of support to prevent the insurgents from making a winning counter-bribe because the insurgents have less flexibility.

We next address the issue illustrated in figure 4.10c: governmental support may decrease as V_I increases. This cannot occur in the continuous case, and we will further analyze this issue via a smaller example. Table 4.4 is analogous to table 4.3, but with four tribes instead of eight.

Tribe Number	Weight $w_5(\theta)$	Preference $v_3(\theta)$
1	0.062	0.000
2	0.188	-0.710
3	0.312	-1.260
4	0.438	-1.710

Table 4.4: Tribe characteristics (Insurgent Advantage Scenario with four tribes)

We investigate the case where $L = 0.4$, and present the results in figure 4.11. In this scenario, the government's support clearly decreases as V_I increases. By inspecting table 4.4, we can see why this is the case. For the government to win, they need to have either tribe 4 or both tribes 2 and 3 in the winning coalition. Tribe 1 is essentially worthless. For the example in table 4.4, there are two possible optimal winning coalitions: tribes 1 and 4, or tribes 1, 2, and

3. In the former case, the government receives intelligence from 0.5 of the population, and in the latter case, the value is 0.562. If the government's coalition contains tribes 1 and 4, then the insurgents need to counter-bribe tribe 4. If the government's coalition is tribes 1, 2, and 3, then the insurgents need to counter-bribe either tribe 2 or 3 (not both), and therefore in this case, the government will need to make tribes 2 and 3 equally costly for the insurgents to counter-bribe. Bribing both tribes 2 and 3 is cost effective for low values of V_I, because it is very costly to convince tribe 4 to support the government. However, as V_I increases eventually (around $V_I = 0.23$) it is too expensive for the government to continue bribing both tribes 2 and 3, and the government switches to optimally bribing only tribe 4. This causes the decrease in Figure 4.11. This illustrates one potential counter-intuitive result that the discrete population model can produce: as the insurgents become stronger the government targets the most extreme pro-insurgent tribe.

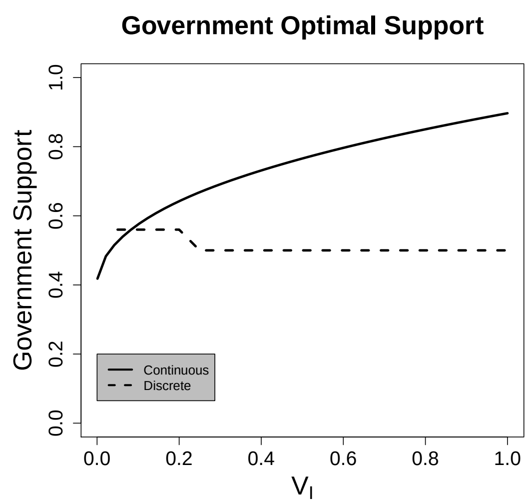


Figure 4.11: Government support level: discrete vs. continuous population with four tribes, $L = 0.4$.

While we did not prove any specific results for the discrete population, Figures 4.9, 4.10 and 4.11 illustrate that the results differ, potentially considerably, from the continuous population. Furthermore, the results are also more sensitive to the weights of the population in the discrete case, and so it is important to expend effort and resources into making accurate estimates of those values.

THIS PAGE INTENTIONALLY LEFT BLANK

CHAPTER 5: Conclusion

The importance of winning the battle for popular support during insurgency conflicts demands that the government have an effective strategy to engage with the population. The interaction with the population is made more complex because the insurgents are an intelligent adversary. If the government does not account for the countermeasures, the insurgents will implement to stifle the flow of intelligence and support to the government, then the government will assuredly lose this battle for the hearts and minds. In this thesis, we make a first attempt to formulate a mathematical model based on political vote-buying models to capture the interactions amongst the government, the insurgents, and the population. The purpose is to provide insight on this topic and provide a base model upon which more complex and realistic analyses can be performed.

The model focuses on three main topics. One is the minimum cost (or equivalently strength) necessary for the government to win this conflict by receiving enough intelligence from the population. The second topic is which individuals in the population will receive bribes from the government and provide the government with intelligence. Finally, we examine the affect of coercion and when it is an effective tool for the insurgents and when it may backfire. We incorporate a weighting function into our model that allows individuals or tribes to have different influence within society.

Many of our results are not surprising. The likelihood the government will win decreases with the strength of the insurgents and the amount of intelligence required for victory. The main results from GS also apply to our model. These results are that government victory results in excess support and that support for the government increases with the strength of the insurgents. We derive additional results that the support for the government increases with the level of support required for victory and the support increases as the population's preferences become weaker. These results follow from the strategic nature of the game. The government must ensure that the bribe it offers will withstand a counter-bribe by the insurgents. Only examining the problem from the government's perspective using a decision analytic framework will lead to different (and incorrect) results and conclusions. Coercion can increase or decrease the likelihood that the government wins. For weak insurgents, it can be an effective tool because they have very little to lose from the collateral damage the coercion may cause. While coercion

may not have a monotonic effect for the insurgents, situational awareness does: the greater the situational awareness of the insurgents, the fewer negative consequences from coercion. If the government requires a large amount of support to achieve victory, then the insurgents should decrease coercion. This follows because it is likely that the government will need support from pro-insurgent individuals and the coercion can only alienate them, making it easier for the government to obtain their support. As the optimal amount of coercion may be some moderate level, an interesting issue from a policy standpoint is whether there are actions that the government can take to entice the insurgents into executing a suboptimal amount of coercion. Finally, if the population consists of a discrete number of tribes that each make a decision to fully support the government or the insurgents, the problem is much more difficult to analyze. Many of the results that apply for a continuous population no longer hold for the discrete population. This issues highlights the importance of assigning weights to tribes. Greedy algorithms will not produce the optimal answer and more complicated algorithms will need to be executed to determine the government's bribe.

Our model has many assumptions and simplifications that we now address. First, we assume that the overall outcome of the interaction is binary: the insurgents win or the government wins. The real outcome is the level of intelligence the government receives. Presumably more intelligence is better for the government. A modified model might define the objective to be the level of intelligence received by the government. The government wants to maximize this quantity and the insurgents want to minimize it. However, this binary assumption may not be that unrealistic as an initial formulation. The government's ability to defeat the insurgents as a function of intelligence may resemble a step function. Furthermore, we only examine the role of popular support in providing intelligence to the government. The population can provide active support to the insurgents in the form of finances, resources, and sanctuary [42]. We feel that focusing initially on the the intelligence piece is valid because the government forces, in most cases, have superior military capabilities and they only need intelligence to effectively target insurgents with force [49].

This model assumes that the net preference function $v(\cdot)$ is known by the insurgents and the government and that bribes can be tailored to each individual. This may be plausible if the population consists of a handful of tribes, but not if it is thousands (or more) of unique individuals. A more realistic model might restrict the government to offering the same bribe to all individuals. Another assumption is the two-period sequential nature of the game. While this does make the analysis much simpler than a simultaneous Colonel Blotto variant, there is some

justification for this. In the sequential game structure the player who moves last has a strategic advantage, and thus neither player wants to move first. If the insurgents are entrenched in an area, they have no incentive to change the status quo. Only after observing the actions of the government will they make a counter-offer. This puts the insurgents in a position of advantage, which is appropriate. Even if the government is in control of the area, they have much more to lose than the insurgents and may be conservative with their strategy. The government may still make the initial offer to prevent an insurgency from growing or taking root in an area. While the sequential component may be reasonable, there will be repeated interactions over time, and possibly learning occurring as there will be incomplete information. Future work should consider multiple interactions.

The model relies on the ability to enforce an individual's decision. If the individual accepts a bribe, it will support that side. In political science, this is referred to as an open ballot. This is a reasonable assumption in our model, because the government observes informants and the insurgents may have some idea about who is helping the government. Our implementation of coercion assumes that the preferences of all individuals are decreased by the same proportion. This was primarily done for analytic convenience, and while it is a reasonable first cut there are other potential formulations (see e.g., [37]). We also assume that the insurgents have to commit to their level of coercion at the beginning of the interaction and cannot change this, which is not realistic. It might be more realistic to assume that the insurgents observe the government's bribe and then the insurgents choose some combination of coercion level and counter-bribe. Future research should consider more appropriate implementations of coercion. Furthermore, we do not assume a cost to implementing coercion. Although we could incorporate a coercion cost, it would complicate the model without providing much additional insight. A further shortcoming of the model is the passive nature of the population. Adding a strategic component to the population would strengthen the model. While we do account for part of the social structure through the weight function, there is still much of the social dynamics we do not model. Tribes or individuals will influence each other via the underlying social network, and this can potentially play a significant role on the amount of intelligence the government receives.

The previous paragraphs list many of the shortcomings of the model and there are many more that we do not list. However, to the best of our knowledge, this thesis is the first attempt to analyze insurgencies with a vote-buying framework. In some sense, this work represents the start more than the finish, as there is much more that could be done. Many potential improvements were listed in the previous paragraphs, so we will not repeat them here. The most immediate

improvement would be to examine the discrete case in more rigorous detail. We have postulated an algorithm that should solve for the government's optimal bribe for a small number of tribes, but it is rather incomplete. More interesting extensions would have the population play an active and strategic part in the interaction. The population benefits from the conflict between the government and insurgents because they receive bribes. However, in practice the population might be able to play the government and the insurgents off each other to extract even more. For example: in the discrete setting, tribes could collude and merge to create a combined tribe that is more attractive than either of the separate tribes. In the current model, the transaction for support does not depend upon the actual outcome of the conflict. In practice, the government could make promises to the population about what the government will provide them if it can defeat the insurgents and can control the given territory. This idea is motivated by the work of [31], which examines the impact of campaign promises in elections.

REFERENCES

[1] D. Galula. *Counterinsurgency Warfare: Theory and Practice*. Praeger Security International, 1964.

[2] M. Montgomery. Iraq: The social context of ieds. *Military Review*, pp. 37–40, 2005.

[3] J.F. Danilowicz. Fish out of water: Applying counter insurgency doctrine in the war on terrorism. Technical report, Naval War Collage, Newport, RI, 2003.

[4] United States Army. *Counterinsurgency*. Headquarters, Department of the Army, Washington, DC, 2006.

[5] T.R. Searle. Tribal engagement in anbar province: The critical role of special operations forces. *Joint Force Quarterly*, 50:62–66, 2008.

[6] T. Groseclose and J.M. Snyder. Buying supermajorities. *The American Political Science Review*, 90(2):303–315, 1996.

[7] E. Dekel, M.O. Jackson, and A. Wolinsky. Vote buying: Legislatures and lobbying. *Quarterly Journal of Political Science*, 4(2):103–128, 2009.

[8] V. Krishna. *Auction Theory*. Academic Press, 2009.

[9] P. Klemperer. Auction theory: A guide to the literature. *Journal of Economic Surveys*, 13(3): 227–286, 1999.

[10] V. Krishna and J Morgan. An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory*, 72(2):343–362, 1997.

[11] Y.K. Che and I. Gale. Expected revenue of all-pay auctions and first-price sealed-bid auctions with budget constraints. *Economics Letters*, 50(3):373–379, 1996.

[12] Y.K. Che and I. Gale. Revenue comparisons for auctions when bidders have arbitrary types. *Theoretical Economics*, 1(1):95–118, 2006.

[13] J.M. Smith. *Evolution and the Theory of Games*. Cambridge University Press, 1982.

[14] D.T. Bishop, C. Cannings, and J.M. Smith. The war of attrition with random rewards. *Journal of Theoretical Biology*, 74(3):377, 1978.

[15] D. Fudenberg and J. Tirole. A theory of exit in duopoly. *Econometrica*, 54(4):943–960, 1986.

[16] B. O'Neill. International escalation and the dollar auction. *Journal of Conflict Resolution*, 30 (1):33, 1986.

[17] P. Dasgupta. The Theory of Technological Competition. In J.E. Stiglitz and G.F. Mathewson, editors, *New Developments in the Analysis of Market Structure*, chapter 17, pp. 519–547. MIT Press, 1986.

[18] G.S. Becker. A theory of competition among pressure groups for political influence. *The Quarterly Journal of Economics*, 98(3):371, 1983.

[19] M.R. Baye, D. Kovenock, and C.G. De Vries. Rigging the lobbying process: an application of the all-pay auction. *The American Economic Review*, 83(1):289–294, 1993.

[20] J. Gans-Morse, S. Mazzuca, and S. Nichter. Varieties of Clientelism: Machine Politics During Elections, 2010. CDDRL Working Paper, Stanford University.

[21] R. Golman and S.E. Page. General Blotto: Games of allocative strategic mismatch. *Public Choice*, 138(3):279–299, 2009.

[22] E Borel. The theory of play and integral equations with skew symmetric kernels. *Econometrica*, 21(1):pp. 97–100, 1953.

[23] O. Gross and R. Wagner. *A Continuous Colonel Blotto Game*. RAND, 1950.

[24] L. Friedman. Game-theory models in the allocation of advertising expenditures. *Operations Research*, 6(5):699–709, 1958.

[25] B. Roberson. The Colonel Blotto game. *Economic Theory*, 29(1):1–24, 2006.

[26] S.M. Chowdhury, D. Kovenock, and R.M. Sheremeta. An Experimental Investigation of Colonel Blotto Games, 2009. APSA 2009 Toronto Meeting Paper.

[27] Y. Gerchak and M. Parlar. Allocating resources to research and development projects in a competitive environment. *IIE Transactions*, 31(9):827–834, 1999.

[28] R. Powell. Sequential, Nonzero-sum Blotto: Allocating Defensive Resources Prior to Attack. *Games and Economic Behavior*, 67(2):611–615, 2009.

[29] Jeffrey S. Banks. Buying supermajorities in finite legislatures. *The American Political Science Review*, 94(3):pp. 677–681, 2000.

[30] C. Dippel. Groseclose and Snyder in Finite Legislatures, 2010. Working Paper, University of Toronto.

[31] E. Dekel, M.O. Jackson, and A. Wolinsky. Vote buying: General elections. *Journal of Political Economy*, 116(2):351–380, 2008.

[32] F.J. Várdy and J. Morgan. On the buyability of voting bodies. *Journal of Theoretical Politics (Forthcoming)*, 2011.

[33] J. Morgan and F. Várdy. Negative vote buying and the secret ballot. *Journal of Law, Economics, and Organization (Forthcoming)*, 2011.

[34] J.C. Heckelman. The effect of the secret ballot on voter turnout rates. *Public Choice*, 82(1): 107–124, 1995.

[35] W.A. Callahan and D. McCargo. Vote-buying in Thailand's northeast: The July 1995 general election. *Asian Survey*, 36(4):376–392, 1996.

[36] P.C. Vicente. Is vote buying effective? Evidence from a field experiment in West Africa. *BREAD and University of Oxford, Working Paper*, 2008.

[37] P. Collier and P.C. Vicente. Violence, bribery, and fraud: The political economy of elections in sub-saharan africa. *Public Choice*, pp. 1–31, 2011.

[38] A. Schedler. The menu of manipulation. *Journal of Democracy*, 13(2):36–50, 2002.

[39] N. Kriger. ZANU (PF) strategies in general elections, 1980–2000: Discourse and coercion. *African Affairs*, 104(414):1, 2005.

[40] G. Gvineria. How does terrorism end? In P. Davis and K. Cragin, editors, *Social Science for Counterterrorism: Putting the Pieces Together*, pp. 257–298. Rand Corp, 2009.

[41] M. Crenshaw. The causes of terrorism. *Comparative politics*, 13(4):379–399, 1981.

[42] C. Paul. How Do Terrorists Generate and Maintain Support? In P. Davis and K. Cragin, editors, *Social Science for Counterterrorism: Putting the Pieces Together*, pp. 113–150. Rand Corp, 2009.

[43] S. Metz and Millen. Insurgency and Counterinsurgency in the 21st Century: Reconceptualizing Threat and Response. Technical report, Army War Collage Strategic Studies Institute, Carlisle Barracks PA., 2004.

[44] T.C. Helmus, C. Paul, and R.W. Glenn. *Enlisting Madison Avenue: The Marketing Approach to Earning Popular Support in Theaters of Operation*. Rand Corporation, 2007.

[45] S.T. Flanigan. Charity as resistance: Connections between charity, contentious politics, and terror. *Studies in Conflict & Terrorism*, 29(7):641–655, 2006.

[46] C. Berrebi. The Economics of Terrorism and Counterterrorism: What Matters and Is Rational-Choice Theory Helpful? In P. Davis and K. Cragin, editors, *Social Science for Counterterrorism: Putting the Pieces Together*, pp. 151–208. Rand Corp, 2009.

[47] B Ganor. *The Counter-Terrorism Pule: A Guide for Decision Makers*. Transaction Publishers, 2005.

[48] H.A. Simon. *Models of Bounded Rationality*, volume 2. MIT Press, Cambridge, MA, 1982.

[49] M. Kress and R. Szechtman. Why defeating insurgencies is hard: the effect of intelligence in counterinsurgency operations, a best case scenario. *Operations Research*, 57:578–585, 2009.

[50] M. Atkinson, M. Kress, and R. Szechtman. Carrots, Sticks and Fog During Insurgencies, 2011. Naval Postgraduate School Working Paper.

[51] G.S. Epstein and I.N. Gang. Who is the enemy. *Defense and Peace Economics*, 18(6):469–484, 2007.

[52] E. Berman, J.N. Shapiro, and J. Felter. Can hearts and minds be bought? The economics of counterinsurgency in Iraq. *NBER Working Paper*, 2008.

[53] United States Army. My Cousin's Enemy is My Friend: A Study of Pashtun "Tribes" in Afghanistan, 2009. Afghanistan Research Reachback Center White Paper: TRADOC G2 Human Terrain System.

[54] J. Gant. *One Tribe at a Time: A Strategy for Success in Afghanistan*. Nine Sisters Imports, Inc, 2009.

[55] A. Giustozzi and N. Ullah. *"Tribes" and Warlords in Southern Afghanistan, 1980-2005*. Crisis States Research Centre, 2006.

[56] Ahmed A.S. Tribes and States in Waziristan. In Richard Tapper, editor, *The Conflict of Tribe and State in Iran and Afghanistan*, pp. 196–197. Croom Helm, 1983.

[57] B. Burgoon. On welfare and terror. *Journal of Conflict Resolution*, 50(2):176, 2006.

[58] A. Long. The Anbar Awakening. *Survival*, 50(2):67–94, 2008.

[59] A.M. Geoffrion. Generalized benders decomposition. *Journal of optimization theory and applications*, 10(4):237–260, 1972.

[60] D. Bertsimas and J. Tsitsklis. *Introduction to Linear Optimization*. Athena Scientific, Belmont, MA, 1997.

[61] J.F. Shapiro. Dynamic programming algorithms for the integer programming problem-I: The integer programming problem viewed as a knapsack type problem. *Operations Research*, 16 (1):103–121, 1968.

APPENDIX

In the Appendix, we provide the proofs and arguments supporting the main results in the thesis. Many of these proofs are somewhat tedious, and where possible we have provided qualitative and intuitive arguments in Section 3 to explain and motivate these technical arguments. In Section A.1, we discuss some of the technical issues with the two-player game under analysis. In Section A.2, we present the primary notation used in Sections A.3 – A.4, which present the analysis for the insurgent and government's strategy, respectively. In Section A.5, we discuss coercion, and finally in Section A.6, we examine the discrete model.

A.1 Technical Considerations

As this is a two player sequential game, we must compute Nash Equilibria (technically subgame perfect Nash Equilibria). As we only consider 2 periods, this can be done via backward induction. We first start with the insurgents' response to the government bribe $b_G(\theta)$ and compute the optimal counter-bribe $b_I^*(\theta)$ as a function of $b_G(\theta)$. This is discussed in Section A.3. Once this optimal response strategy is determined, we can calculate the government's optimal initial bribe $b_G^*(\theta)$, which must account for the insurgents' optimal counter-bribe. We examine the government's bribe in Section A.4. The strategies of the government and insurgent are functions $b_G(\theta)$ and $b_I(\theta)$ on the domain of individuals $\theta \in [-\frac{1}{2}, \frac{1}{2}]$. As the strategy space is not a finite or discrete set, we mention some technical issues that may arise in the analysis.

For any bribe $b_G(\theta)$ there are an infinite number of equivalent bribes that differ from $b_G(\theta)$ on a set of measure 0. This issue is not important for our purposes and we will ignore it. Furthermore, because the insurgents move second there are many potential "losing" counter-bribes that are technically equilibria. Individuals will only accept the bribe from the party that offers the more compelling bribe. Thus, if the insurgents realize they will lose, then it is optimal for them to offer a counter-bribe that will be of zero cost. Obviously, the bribe $b_I(\theta) = 0$ satisfies this condition. However, technically, other counter-bribes that are less compelling than the government's bribe could also be offered, knowing that they would be rejected by the population. These counter-bribes would also be optimal counter-bribes by the insurgents, because they do not have to actually execute the counter-bribes and so are zero cost. We also ignore these strategies and just focus on the interesting bribe of zero $b_I(\theta) = 0$. We can avoid these meaningless bribes by making an assumption that offering a non-zero bribe has some

negligible cost, and this will result in the unique counter-bribe of $b_I^*(\theta) = 0$ when the insurgents want to concede.

This assumption of assuming a negligible cost to offering a bribe is also useful for dealing with ties. We assume that the government will only execute a bribe if the total cost is less than V_G and, similarly, the insurgents will only execute a bribe if it costs less than V_I. If the bribe costs exactly V_G (or V_I), the bribe will not be offered because of this negligible extra cost. This allows us to assume that an optimal strategy for the government will be a bribe that will cost the insurgents exactly V_I to counteract, and hence the insurgents will not make a counter-offer. If the government offers a bribe that costs the insurgents less than V_I to successfully counter-bribe, then it will be insufficient to deter the insurgents; if it costs the insurgents more than V_I to win the government is incurring unnecessary cost with its original bribe. Finally, we assume that if an individual is indifferent between the government and insurgents (i.e., $v(\theta) + b_I(\theta) = b_G(\theta)$), they will support the insurgents. Thus in practice if the insurgents make a positive bribe to an individual θ the bribe will satisfy $b_I(\theta) = b_G(\theta) - v(\theta)$.

A.2 Notation

This section introduces notation used in this Appendix (primarily in Sections A.3 – A.4) and describes the qualitative meaning behind some of the parameters. Suppose that the government makes a bribe of $b_G(\theta)$. Define

$$\tilde{v}(\theta) = v(\theta) + b_G(\theta) \tag{A.1a}$$

$$S_G = \int_{\tilde{v}(\theta)>0} w(\theta)d\theta \tag{A.1b}$$

$$B_G = \{\theta \ : \ b_G(\theta) > 0\} \tag{A.1c}$$

$$A_L^T = \{\theta \ : \ 0 < \tilde{v}(\theta) < T\} \tag{A.1d}$$

$$A_E^T = \{\theta \ : \ \tilde{v}(\theta) = T\} \tag{A.1e}$$

$$A^T = A_L^T \cup A_E^T \tag{A.1f}$$

$$S_G^T = \int_{A_L^T} w(\theta)d\theta \tag{A.1g}$$

$$T^*(K) = \inf\{T \ : \ \int_{A^T} w(\theta)d\theta \geq K\}, \quad K \leq S_G \tag{A.1h}$$

$$A(K) \in \{\hat{A} \subset A_E^{T^*(K)} \ : \ \int_{\hat{A}} w(\theta)d\theta = K - \int_{A_L^{T^*(K)}} w(\theta)d\theta\} \tag{A.1i}$$

We now give the qualitative description of these terms. The function $\tilde{v}(\theta)$ is the population's utility if they accept the government's bribe and support the government. The quantity S_G is the fraction of the population that supports the government under the government's bribe $b_G(\theta)$ without a counteroffer from the insurgents. Similarly, the support S_G^T is the amount of the government's supporters with preferences strictly less than T. B_G is the set of individuals that receive a bribe from the government. The quantity A_L^T is the set of individuals in the population whose utility from accepting the government's bribe of $b_G(\theta)$ is strictly less than T. The analogous quantity A_E^T is the set of individuals in the population whose utility is exactly T. $T^*(K)$ is the smallest threshold, such that the insurgents will gain at least a fraction K of the government's supporters if the insurgents make a successful counter-bribe to all government supporters with utility less than or equal to this threshold. The values A_L^T, A_E^T, $T^*(K)$ are primarily used in Section A.3 to derive the insurgents' counter-bribe.

If A_E^T has positive mass, the insurgents may not need to counter-bribe all individuals in A_E^T to

win. This is the motivation for the value $A(K)$. If $\int_{A^{T^*(K)}} w(\theta)d\theta > K$, then the insurgents only need to bribe $A(K) \subset A_E^{T^*(K)}$ to obtain the necessary mass of K. Since all individuals in the set $A_E^{T^*(K)}$ require a counter-bribe of $T^*(K)$ from the insurgents, the subset $A(K)$ will not be unique (in general there will be an infinite number of these subsets). The insurgents need to choose the subset $A(K)$ so that the total mass they obtain from A_L^T and $A(K)$ is K.

A.3 Insurgents' Strategy

In this section, we present the proofs for the propositions in section 3.2.2. We first prove Proposition 1 by proving Lemma 1. The threshold defined in Proposition 1 corresponds to $T^*(S_G - L)$ in equation (A.1h), the sets defined in Proposition 1 correspond to $A_L^{T^*(S_G-L)}$ (equation (A.1d)) and $A_E^{T^*(S_G-L)}$ (equation (A.1e)), and the subset described in Proposition 1 corresponds to $A(S_G - L)$ (equation (A.1i))

Lemma 1. *Suppose the government makes an initial bribe $b_G(\theta)$. A lowest cost bribe for the insurgents to capture $0 \leq K \leq S_G$ of the government's supporter is to give a payment of $b_I(\theta) = \tilde{v}(\theta)$ for $\theta \in A_L^{T^*(K)} \cup A(K)$ and a bribe $b_I(\theta) = 0$ for $\theta \notin A_L^{T^*(K)} \cup A(K)$. If $\int_{A_E^{T^*(K)}} w(\theta)d\theta = 0$ this bribe is unique. The quantities $A_L^{T^*(K)}$, $A(K)$, $A_E^{T^*(K)}$ are defined in section A.2.*

Proof. Suppose that the insurgents' low cost counter-bribe involves bribes to the set B. First, it must be that $\tilde{v}(\theta) > 0$ for all $\theta \in B$ because making payments to supporters of the insurgents cannot decrease the level of support for the government. Second, if the insurgents must convince K of the government's supporter to switch allegiances then $\int_B w(\theta)d\theta \geq K$. The set $A_L^{T^*(K)} \cup A(K)$ satisfies these two conditions by construction. To show $A_L^{T^*(K)} \cup A(K)$ is a lowest cost set of individuals the insurgents can bribe to obtain K, we first show that a set B such $B \not\subset A^{T^*(K)}$ cannot be lower cost. To do this, first assume that there is another set B such that $B \not\subset A^{T^*(K)}$, $\tilde{v}(\theta) > 0$ for all $\theta \in B$, and

$$\int_B w(\theta)d\theta = K \tag{A.2a}$$

$$\int_B \tilde{v}(\theta)w(\theta)d\theta < \int_{A_L^{T^*(K)} \cup A(K)} \tilde{v}(\theta)w(\theta)d\theta, \tag{A.2b}$$

where condition (A.2b) states that it costs less to bribe the set B than $A_L^{T^*(K)} \cup A(K)$. We partition B into two subsets $B_1 = B \cap (A_L^{T^*(K)} \cup A(K))$ and $B_2 = B \setminus B_1$. We can also

partition $A_L^{T^*(K)} \cup A(K)$ into B_1 and $A_1 = (A_L^{T^*(K)} \cup A(K)) \setminus B_1$. By condition (A.2b)

$$\int_B \tilde{v}(\theta)w(\theta)d\theta < \int_{A_L^{T^*(K)} \cup A(K)} \tilde{v}(\theta)w(\theta)d\theta \tag{A.3a}$$

$$\int_{B_1} \tilde{v}(\theta)w(\theta)d\theta + \int_{B_2} \tilde{v}(\theta)w(\theta)d\theta < \int_{B_1} \tilde{v}(\theta)w(\theta)d\theta + \int_{A_1} \tilde{v}(\theta)w(\theta)d\theta \tag{A.3b}$$

$$\int_{B_2} \tilde{v}(\theta)w(\theta)d\theta < \int_{A_1} \tilde{v}(\theta)w(\theta)d\theta \tag{A.3c}$$

Furthermore it must hold that $\int_{B_2} w(\theta)d\theta = \int_{A_1} w(\theta)d\theta$, since $\int_B w(\theta)d\theta = \int_{A_L^{T^*(K)} \cup A(K)} w(\theta)d\theta = K$. However by construction $\inf_{\theta \in B_2} \tilde{v}(\theta) \geq T^*(K) \geq \sup_{\theta \in A_1} \tilde{v}(\theta)$. Thus by monotonicity of expectations

$$\int_{B_2} \tilde{v}(\theta)w(\theta)d\theta \geq \int_{A_1} \tilde{v}(\theta)w(\theta)d\theta \tag{A.4}$$

Which is a contradiction. Therefore the lowest cost set B of individuals must satisfy $B \subset A^{T^*(K)}$. An analogous argument shows that the insurgents will first bribe all the individuals in $A_L^{T^*(K)}$ and if necessary a subset $A(K) \subset A_E^{T^*(K)}$ to reach the necessary mass of K. This holds because $\inf_{\theta \in A_E^{T^*(K)}} \tilde{v}(\theta) = T^*(K) \geq \sup_{\theta \in A_L^{T^*(K)}} \tilde{v}(\theta)$. Thus, the most cost effective way to gain the support of K of the government's supporters is to only bribe individuals in $A^{T^*}(K)$. The subset $A(K)$ may not be unique, and thus neither will the optimal insurgent bribe. However, if $\int_{A_E^{T^*(K)}} w(\theta)d\theta = 0$, then the set to bribe will be the unique set $A_L^{T^*(K)}$ because $A(K)$ is a measure 0 set. $\square$

Lemma 1 describes a lowest cost counter-bribe by the insurgents. In the next proposition, which corresponds to Proposition 2 in Section 3.2.2, we summarize when the insurgents should execute this bribe. The set D in Proposition 2 in Section 3.2.2 corresponds to $A_L^{T^*(K)} \cup A(K)$.

Proposition 14. *Suppose the government makes a bid $b_G(\theta)$ then a best response of the insur-*

gents is to bid $b_I(\theta)$ given by

$$b_I(\theta) = 0q \quad \textit{if } T^*(S_G - L) = 0 \tag{A.5a}$$

$$b_I(\theta) = 0q \quad \textit{if } T^*(S_G - L) > 0 \textit{ and } \int_{A_L^{T^*(K)} \cup A(K)} \tilde{v}(\theta)w(\theta)d\theta > V_I \tag{A.5b}$$

$$\begin{aligned} b_I(\theta) = &\tilde{v}(\theta)q \quad \textit{for } \theta \in A_L^{T^*(K)} \cup A(K) \\ &\textit{if } \tilde{T}^*(S_G - L) > 0 \textit{ and } \int_{A_L^{T^*(K)} \cup A(K)} \tilde{v}(\theta)w(\theta)d\theta < V_I \end{aligned} \tag{A.5c}$$

Proof. The expression (A.5a) corresponds to the government not having enough support to win. If $T^*(S_G - L) = 0$ then $S_G < L$, and thus the insurgents do not need to make any bribes to win. The expression (A.5b) states that the government has enough support to win absent insurgent support ($T^*(S_G - L) > 0$). However, by Lemma 1, the cheapest way for the insurgents to gain $S_G - L$ of the government's supporters is to bribe a set $A_L^{T^*(K)} \cup A(K)$. By assumption in expression (A.5b), the cost to bribe these individuals $\int_{A_L^{T^*(K)} \cup A(K)} \tilde{v}(\theta)w(\theta)d\theta$ is worth more than the insurgents value the victory. Thus the insurgents do not make a counter-bribe and concede the victory to the government. The final case in expression (A.5c) is the same as case in expression (A.5b), but the insurgents now value the victory (V_I) more than it will cost to attain it. Thus the insurgents make their most cost effective bribe to the population (see Lemma 1) by setting $b_I(\theta) = \tilde{v}(\theta)$ for $\theta \in A_L^{T^*(K)} \cup A(K)$ and $b_I(\theta) = 0$ for $\theta \notin A_L^{T^*(K)} \cup A(K)$. In this case the insurgents win. □

Finally, an optimal bribe by the government can always be formulated as a leveling bribe (see Proposition 3 of Section 3.2.3 in Section A.4.1) where $\tilde{v}(\theta)$ is nonincreasing. Corollary 2 defines the insurgents' optimal response for that case.

Corollary 2. *If the government makes a bid $b_G(\theta)$ such that the function $\tilde{v}(\theta)$ is nonincreasing then a best response of the insurgents is to bid $b_I(\theta)$ given by*

$$b_I(\theta) = 0q \quad \textit{if } \theta_N < \theta_L \tag{A.6a}$$

$$b_I(\theta) = 0q \quad \textit{if } \theta_N \geq \theta_L \textit{ and } \int_{\theta_L}^{\theta_N} \tilde{v}(\theta)w(\theta)d\theta > V_I \tag{A.6b}$$

$$\begin{aligned} b_I(\theta) = \tilde{v}(\theta)q &\quad \textit{for } \theta \in [\theta_L, \theta_N] \\ q &\quad \textit{if } \theta_N \geq \theta_L \textit{ and } \int_{\theta_L}^{\theta_N} \tilde{v}(\theta)w(\theta)d\theta < V_I \end{aligned} \tag{A.6c}$$

where $\theta_N = \tilde{v}^{-1}(0)$*,*

Proof. If $\tilde{v}(\theta)$ is nonincreasing, then $T^*(S_G - L) = \max(\tilde{v}(\theta_L), 0)$ and $A_L^{T^*(K)} \cup A(K) = [\theta_L, \theta_N]$. Equations (A.6a)–(A.6c) then follow immediately from (A.5a)–(A.5c). Recall by Lemma 1 that this optimal bid schedule is unique only if $\int_{\theta\,:\,\tilde{v}(\theta)=v(\theta_L)} w(\theta)d\theta = 0$.

□

A.4 Government's Strategy

In this section, we examine the government's strategy and prove the results from Section 3.2.3. Before presenting the analysis, we define what a *dominated strategy* is in this context. Obviously, the government will never consider a dominated strategy.

Definition 9. ***Dominated strategy****:* $b_G(\theta)$ *is dominated by a bribe* $b'_G(\theta)$ *if one of the two conditions holds*

- *The cost for the government to execute* $b_G(\theta)$ *is more than the cost to execute* $b'_G(\theta)$*, and the lowest-cost winning insurgent counter-bribe to* $b_G(\theta)$ *costs no more to implement than the corresponding counter-bribe to* $b'_G(\theta)$*.*
- *The lowest-cost winning insurgent counter-bribe to* $b_G(\theta)$ *costs less to implement than the corresponding counter-bribe to* $b'_G(\theta)$*, and the cost for the government to execute* $b_G(\theta)$ *is no less than the cost to execute* $b'_G(\theta)$*.*

Definition 9 states a bribe by the government is dominated if another bribe exists that costs less for the government to implement and costs more for the insurgents to defeat. Before examining the specific nature of the government's optimal bribe in Section A.4.2, we first prove that an optimal strategy for the government is a leveling strategy in Section A.4.1. We discuss the low cost bribe in Section A.4.3 and how the the government's level of support varies with the model parameters in Sections A.4.4–A.4.5.

A.4.1 Leveling Strategy

Here we show a leveling strategy is optimal. We first present Lemmas 2 and 3 which show that the government's optimal strategy can always be constructed such that the set of individuals the government bribes is an interval on $[-\frac{1}{2}, \frac{1}{2}]$.

Lemma 2. *If the government makes an optimal bribe $b_G^*(\theta)$ then the set of the individuals bribed by the government must satisfy $B_G \subset A^{T^*(S_G-L)}$.*

Proof. If the government makes an optimal bribe and $B_G \not\subset A^{T^*(S_G-L)}$, this implies there is some set $C \subset B_G$ such that $\tilde{v}(\theta) > T^*(S_G - L)$ for all $\theta \in C$. Thus, the government could decrease its bribe of the individuals in C by some ϵ (possibly depending upon θ) and these individuals would still not be in $A^{T^*(S_G-L)}$, and hence would not effect the potential counter-bribe of the insurgents. This modified bribe would be less costly than the original bribe, and thus the original bribe could not have been optimal since it was a dominated strategy (see Definition 9). □

Lemma 3. *If the government makes an optimal bribe $b_G^*(\theta)$ to the population, then the set of the individuals bribed by the government can always be constructed as a range of individuals in the population: $B_G = [\underline{\theta}, \bar{\theta}]$ for some $\underline{\theta}$ and $\bar{\theta}$.*

Proof. We first assume the government makes an optimal bribe $b_G^*(\theta)$ such that $\underline{\theta} = \inf_\theta B_G$, $\bar{\theta} = \sup_\theta B_G$ and there is some $C \subset [\underline{\theta}, \bar{\theta}]$ such that $C \cap B_G = \emptyset$. We now show a modified bribe that does bribe the individuals in C cannot perform worse than $b_G^*(\theta)$.

We further assume there is some $\kappa > 0$ such that $[\underline{\theta}, \underline{\theta}+\kappa] \subset B_G$. This rules out the government making a measure zero point bribe to individual $\underline{\theta}$ without making a bribe to any of her neighbors in a sufficiently small neighborhood. If this assumption does not hold, then because this is a measure zero bribe we can set $b_G^*(\underline{\theta}) = 0$ and remove this individual from B_G and compute a new $\underline{\theta}$. We now define $\theta_C = \inf_\theta C$. By construction $\theta_C > \underline{\theta} + \kappa$. Because we assume $v(\theta)$ is continuous and decreasing and $[\underline{\theta}, \underline{\theta} + \kappa] \subset B_G$, then

$$\inf_{\theta \in [\underline{\theta}, \underline{\theta}+\kappa]} \tilde{v}(\theta) > \tilde{v}(\theta_C) = v(\theta_C). \tag{A.7}$$

By Lemma 2 $B_G \subset A^{T^*(S_G-L)}$ and thus $T^*(S_G - L) \geq \sup_{\theta \in [\underline{\theta}, \underline{\theta}+\kappa]} \tilde{v}(\theta) > v(\theta_C)$. The last inequality following from equation(A.7). We will now choose a δ and ϵ and define a modified bribe $b'_G(\theta)$;

$$b'_G(\theta) = \begin{cases} b_G^*(\theta) - \delta & \text{for } \theta \in B_G \\ \epsilon & \text{for } \theta \in C. \end{cases} \tag{A.8}$$

We will show that for the right choice of δ and ϵ, the new bribe schedule $b'_G(\theta)$ defined by equation (A.8) is a better bribe than $b^*_G(\theta)$ and hence $b^*_G(\theta)$ cannot be optimal. That is, $b'_G(\theta)$ weakly dominates $b^*_G(\theta)$ (see Definition 9). To avoid confusion. we will use $T^*(S_G - L)$ to denote the threshold associated with the bribe $b^*_G(\theta)$ and $T'(S_G - L)$ to denote the threshold associated with the bribe $b'_G(\theta)$. First, we will choose δ and ϵ such that

$$\delta \int_{B_G} w(\theta) d\theta = \epsilon \int_C w(\theta) d\theta \tag{A.9}$$

This ensures the new bribe $b'_G(\theta)$ is the same cost for the government to implement as $b^*_G(\theta)$. By Lemma 2 and condition (A.7), $v(\theta_C) < T^*(S_G - L)$ for the original bribe $b^*_G(\theta)$, and thus $C \subset A_L^{T^*(S_G-L)}$ for $b^*_G(\theta)$. By Lemma 1, this means that the insurgents would bribe all of C in a lowest cost counter-bribe to $b^*_G(\theta)$. If we set δ and ϵ small enough such that $\inf_{\theta \in [\underline{\theta}, \underline{\theta}+\kappa]}(\tilde{v}(\theta) - \delta) > v(\theta_C) + \epsilon$ (which is possible by condition (A.7)), then $v(\theta_C) + \epsilon < T'(S_G - L)$ for the modified bribe $b'_G(\theta)$. Thus, $C \subset A_L^{T'(S_G-L)}$ for the new bribe $b'_G(\theta)$ and the cost to the insurgents to bribe the individuals in C in a lowest cost counteroffer is $\epsilon \int_C w(\theta) d\theta$. If the insurgents need to bribe all of B_G as well then their savings will be $\delta \int_{B_G} w(\theta) d\theta$ and the equality in (A.9) holds for the insurgents. However, the insurgents do not necessarily need to make a counter-bribe to everyone in B_G to win. Lemma 2 only states that the individuals in B_G will be under consideration for a bribe (i.e., $A^{T^*(S_G-L)}$), not that they would necessarily receive one (which would be $A_L^{T^*(S_G-L)} \cup A(S_G - L)$ by Lemma 1). The insurgents only need to bribe a subset of B_G that has mass $S_G - L - \int_C w(\theta) d\theta \leq \int_{B_G} w(\theta) d\theta$. The term on the left hand side of the preceding inequality follows because the insurgents will always bribe C by construction. Putting this inequality together with equation (A.9) yields

$$\left(S_G - L - \int_C w(\theta) d\theta \right) \delta \leq \epsilon \int_C w(\theta) d\theta \tag{A.10}$$

The left hand side of (A.10) is the decrease in the cost the insurgents' low cost counter-bribe as a result of $b'_G(\theta)$ and the right hand side is the additional cost to increase the bribe to C. Thus, if the government implements $b'_G(\theta)$, the lowest cost counter-bribe by the insurgents cannot be smaller than the one determined by $b^*_G(\theta)$. Therefore an optimal bribe can always be formulated as $B_G = [\underline{\theta}, \bar{\theta}]$.

□

Below we present the proof for Proposition 3 of Section 3.2.3. This proposition states that the bribe for the insurgents can always be constructed as a leveling bribe. This allows us to focus only on those types of bribing strategies for the government.

Proof. We define $v^{-1}(a) = -\frac{1}{2}$ for $a > v(-\frac{1}{2})$. First, we will show that an optimal bribe $b_G(\theta)$ exists such that $\tilde{v}(\theta) = a$ for $\theta \in B_G$. That is, all individuals that are bribed by the government are bribed such that they are equally expensive for the insurgents to counter-bribe. The argument is similar to that in Lemma 3. We assume that there is some optimal bribe $b_G^*(\theta)$ such that $\tilde{v}(\theta)$ is not constant for $\theta \in B_G$. For notational simplicity, let us define $T = T^*(S_G - L)$ from equation (A.1h) to be the threshold associated with the bribe $b_G^*(\theta)$. First, by Lemma 2 $B_G \subset A^T$ and thus $\tilde{v}(\theta) \leq T$ for $\theta \in B_G$. Since $\tilde{v}(\theta)$ is not constant, this implies there exist some set $C \subset B_G$ such that $\sup_{\theta \in C} \tilde{v}(\theta) < T - \kappa$ for some small κ. We will derive a contradiction by showing that for an optimal bribe $\tilde{v}(\theta) \geq T$ for $\theta \in B_G$.

The argument follows the same mechanics as in Lemma 3 so we will just sketch the argument. For a given δ and ϵ, we modify the bribe such that $b'_G(\theta) = b_G^*(\theta) - \delta$ for $B_G \setminus C$ and $b'_G(\theta) = b_G^*(\theta) + \epsilon$ for C. The values of δ and ϵ must satisfy

$$\delta \int_{B_G \setminus C} w(\theta) d\theta = \epsilon \int_C w(\theta) d\theta \tag{A.11}$$

This ensures that it costs the government the same amount to implement $b_G^*(\theta)$ and $b'_G(\theta)$. Because $\sup_{\theta \in C} \tilde{v}(\theta) < T$, under $b_G^*(\theta)$ we have $C \subset A_L^T$ and thus by Lemma 1 the insurgents would target C in any counter-bribe to $b_G^*(\theta)$. We now define T' to be the threshold from equation (A.1h) associated with the modified bribe $b'_G(\theta)$. If δ and ϵ are small enough, then $\sup_{\theta \in C} \tilde{v}(\theta) < T' - \epsilon$ and thus $C \subset A_L^{T'}$ and the insurgents would also C target in any lowest cost response to the modified bribe $b'_G(\theta)$. This would increase the lowest cost response of the insurgents by $\epsilon \int_C w(\theta) d\theta$. As in Lemma 3, the insurgents will not necessarily need to bribe everyone in $B_G \setminus C$ to win, and thus the lowest cost bribe of the insurgents for the modified bribe $b'_G(\theta)$ would decrease by at most $\delta \int_{B_G \setminus C} w(\theta) d\theta$. Thus, the modified bribe $b'_G(\theta)$ weakly dominates the original bribe $b_G^*(\theta)$, because this bribe is the same cost to the government and cannot decrease the cost to the insurgents to respond (see Definition 9). This proves that an optimal bribe given by the government can be constructed as a leveling bribe such that $\tilde{v}(\theta)$ is flat on B_G.

We will define $\tilde{v}(\theta) = a$ on B_G for some a. From Lemma 3, we know B_G is an interval, which

we define to be $[d, m]$. We now show that $d = v^{-1}(a)$. Because $v(\theta)$ is decreasing, $b^*_G(\theta)$ is positive, and $\tilde{v}(\theta) = a$ on $[d, m]$ it must be that $a \geq v(d)$. If we assume that $a = v(d) + \beta$ for some $\beta > 0$, then we can define the set $E = [v^{-1}(a - \frac{\beta}{2}), d]$. By construction, E receives no bribes from the government and

$$\sup_{\theta \in E} \tilde{v}(\theta) = \sup_{\theta \in E} v(\theta) = a - \frac{\beta}{2} < \sup_{\theta \in B_G} \tilde{v}(\theta) = a. \tag{A.12}$$

Thus by Lemma 1, the insurgents would bribe the members of E in a lowest cost counter-bribe. However, now we can make the same δ/ϵ argument to reduce the bribe in B_G by δ and increase the bribe in E by ϵ, and this will result in a modified bribe that weakly dominates the original bribe. Therefore, we can always construct a leveling bribe of height a such that $d = v^{-1}(a)$. Clearly, if $a > v(-\frac{1}{2})$, then there is a leveling bribe on the interval $[-\frac{1}{2}, m]$ and hence the logical definition $v^{-1}(a) = -\frac{1}{2}$ for $a > v(-\frac{1}{2})$.

Finally, if $\theta_L > \theta_N$, then in an optimal bribe $m \geq \theta_L > \theta_N$. If $\theta_L \leq \theta_N$, then it is not optimal for $m < \theta_N$. If $m < \theta_N$ then we can construct a δ/ϵ argument with B_G and $C = [m, \theta_N]$ and reduce the bribes in B_G by δ and bribe the individuals C an amount ϵ. The cost to the government would remain the same, but the cost to the insurgents would weakly increase.

□

A.4.2 Government Bribe

We now prove Proposition 4 in Section 3.2.3. This presents the general form of the government's low cost bribe, assuming that V_G is large enough for the government to win.

Proof. We look at each of the four cases separately.

Case 1

Assume $\theta_L < \theta_N$ and $V_I \leq \int_{\theta_L}^{\theta_N} v(\theta)w(\theta)d\theta$. If the government offers no bribe to the population (i.e., $b_G(\theta) = 0$), then the insurgents need to buy off a fraction $W(\theta_N) - L$ of the pro-government population. By Corollary 2, the lowest cost way to do this is to bribe the individuals on the interval $\theta \in [\theta_L, \theta_N]$. This would cost $\int_{\theta_L}^{\theta_N} v(\theta)w(\theta)d\theta$, which is larger than the value of the victory to the insurgents, V_I. Thus, the insurgents do attempt to bribe any individuals because the cost of victory is too high. Anticipating this, the government makes no offers,

so $b_G^*(\theta) = 0$.

Case 2

Suppose that $\theta_L < \theta_N$ and $\int_{\theta_L}^{\theta_N} v(\theta)w(\theta)d\theta \leq V_I \leq v(\theta_L)\int_{\theta_L}^{\theta_N} w(\theta)d\theta$. First, the government will never make an offer to any individuals with $\theta < \theta_L$. The logic follows from from Lemma 2. For the government to bribe individuals with $\theta < \theta_L$ would require those individuals to be in $A^{T^*(S_G - L)}$. This can only occur if $v(\theta) + b_G(\theta) > v(\theta_L)$ for all $\theta \in [\theta_L, \theta_N]$. However, this implies the government's bribes are unnecessarily large because $V_I \leq v(\theta_L)\int_{\theta_L}^{\theta_N} w(\theta)d\theta$ and thus the government could decrease its bribes for $\theta \in [\theta_L, \theta_N]$ and still defeat the insurgents.

Next, we show that the government will never bribe any individuals $\theta > \theta_N$. If the government only bribes the individuals in $\theta \in [\theta_L, \theta_N]$ such that $v(\theta) + b_G(\theta) \leq v(\theta_L)$, then it must hold that that $\int_{\theta_L}^{\theta_N}(v(\theta) + b_G(\theta))w(\theta)d\theta = V_I$, because that is the cost of the lowest cost counter-bribe by the insurgents. The total cost to the government to execute the bribe will be

$$\int_{\theta_L}^{\theta_N} b_G(\theta)w(\theta)d\theta = V_I - \int_{\theta_L}^{\theta_N} v(\theta)w(\theta)d\theta \tag{A.13}$$

If instead the government bribes a certain set $B \subset [\theta_L, \theta_N]$ and $C \subset [\theta_N, \frac{1}{2}]$, then we have the the cost of bribes being

$$\int_{B \cup C} b_G(\theta)w(\theta)d\theta = V_I - \int_B v(\theta)w(\theta)d\theta + \int_C -v(\theta)w(\theta)d\theta \tag{A.14}$$

Because $v(\theta) < 0$ for $\theta \in C$, the right hand side of (A.14) is larger than the right hand side of (A.13), and so it will cost the government more to bribe individuals $\theta > \theta_N$.

We have shown that the government will only bribe individuals for $\theta \in [\theta_L, \theta_N]$. Equation (A.13) specifies the minimum cost of the optimal bribe. There are many potential bribes that will satisfy the condition. However, by Proposition 3 of Section 3.2.3 there exists an optimal leveling bribe, which is the one we will focus. By continuity of $w(\theta)$ and the assumption of this case, there exists a value $a \in (0, v(\theta_L)]$ that satisfies

$$\int_{\theta_L}^{v^{-1}(a)} v(\theta)w(\theta)d\theta + a\int_{v^{-1}(a)}^{\theta_N} w(\theta) = V_I \tag{A.15}$$

Once we compute this a, then the bribe $b_G(\theta) = a - v(\theta)$ for $\theta \in [v^{-1}(a), \theta_N]$ satisfies equa-

tion (A.13). To see why, we can compute such an a, we define the left hand side of equation (A.15) as $f(a)$. By the assumption of this case, $f(0) \leq V_I \leq f(v(\theta_L))$. Furthermore, $f'(a) = \int_{v^{-1}(a)}^{\theta_N} w(\theta) \geq 0$ and thus by the intermediate value theorem there is a unique solution a to (A.15) and hence a unique optimal leveling bribe for the government.

Case 3

By Proposition 3 of Section 3.2.3, the government's optimal bribe $b_G^*(\theta)$ can always be constructed as a leveling bribe such that $b_G^*(\theta)$ satisfies $\tilde{v}(\theta) = v(\theta) + b_G^*(\theta) = a$ for $\theta \in [v^{-1}(a), m]$. Because this bribe produces a nonincreasing function $\tilde{v}(\theta)$, by Corollary 2 the lowest cost counter-bribe will cost $a \int_{\theta_L}^{m} w(\theta)d\theta = a(S_G(m) - L)$. Therefore, the government's lowest cost bribe must set this cost to V_I to deter the insurgents from making a counteroffer. Thus, to ensure the victory the government needs to set.

$$a = \frac{V_I}{S_G(m) - L} \tag{A.16}$$

The preceding calculations assumed that $v^{-1}(a) \leq \theta_L$ or $a \geq v(\theta_L)$. We will now show that must be true. If we assume that $a < v(\theta_L)$, then we can make a δ/ϵ argument similar to those used in Lemma 3 and Proposition 3 of Section 3.2.3 to show this is not an optimal bribe by the government. To do this, we look at a modified bribe schedule $b'_G(\theta)$ that will reduce m by δ and increase a by ϵ. We will choose ϵ and δ, such that the cost for the insurgents to make a successful counter-bribe would cost the same under $b_G^*(\theta)$ and $b'_G(\theta)$. Note that under $b'_G(\theta)$, the insurgents would target individuals for $\theta \in [\theta_L, m - \delta]$ in a counter-bribe for ϵ and δ small enough. We will choose δ and ϵ to satisfy

$$a \int_{m-\delta}^{m} w(\theta)d\theta = \epsilon \int_{v^{-1}(a)}^{m-\delta} w(\theta)d\theta + \int_{v^{-1}(a+\epsilon)}^{v^{-1}(a)} (a + \epsilon - v(\theta))w(\theta)d\theta \tag{A.17}$$

The value on the left-hand side is the reduction in the lowest cost counter-bribe of the insurgents now that that they do not have to bribe individuals in $\theta \in [m - \delta, m]$, and the right hand side is the increase in the cost to the insurgents to bribe the individuals $\theta \in [v^{-1}(a + \epsilon), m - \delta]$. A δ/ϵ combination can always be calculated to satisfy equation (A.17), and they can be made small enough such that the lowest cost insurgent counter-bribe will still only target individuals $\theta \in [\theta_L, m - \delta]$. By construction of (A.17), the insurgents will pay the same amount to optimally respond to $b'_G(\theta)$ as they would to the original bribe $b_G^*(\theta)$. However, the government will pay

less for this modified bribe. First, in this scenario $m > \theta_N$. This follows because in this case we are assuming $a < v(\theta_L)$ and we know that $V_I > v(\theta_L) \int_{\theta_L}^{\theta_N} w(\theta)d\theta$, so the insurgents would win if $m = \theta_N$. Therefore, in this scenario an optimal bribe must involve $m > \theta_N$. Next the government must pay an additional amount equal to the right hand side of (A.17) according to this modified bribe $b'_G(\theta)$. However, the amount they save by not having to bribe $\theta \in [m-\delta, m]$ is $\int_{m-\delta}^{m}(a - v(\theta))w(\theta)d\theta$. Because $m > \theta_N$, δ can be chosen such that $m - \delta > \theta_N$ and so $\int_{m-\delta}^{m} -v(\theta)w(\theta)d\theta > 0$ and the government's reduction in cost to implement $b'_G(\theta)$ is more than the left hand side of (A.17). Thus the modified bribe $b'_G(\theta)$ costs less to implement than $b^*_G(\theta)$, and the modified bribe dominates the original (see Definition 9). Therefore,

$$a = \frac{V_I}{S_G(m) - L} \geq v(\theta_L), \tag{A.18}$$

and equation (A.16) is valid and defines the optimal leveling bribe. Finally by Proposition 3 of Section 3.2.3, $m \geq \theta_N$.

Case 4

The logic is exactly the same as the beginning of Case 3. The government will implement a leveling bribe such that $\tilde{v}(\theta) = v(\theta) + b_G(\theta) = a$ on some interval $\theta \in [v^{-1}(a), m]$. The value of a satisfies

$$a = \frac{V_I}{S_G(m) - L} \tag{A.19}$$

We do not need to justify $a \geq v(\theta_L)$ because $v(\theta_L) < 0$ in this case. We know that $m > \theta_L$ by Proposition 6.

□

A.4.3 Government's Low Cost Bribe

Here we prove Prop 5 in Section 3.2.3 by calculating m^* .

Proof. The height of the leveling bribe $H(m)$ is defined in equation (3.6) in Section 3.2.3. We can define its inverse because $v(\theta)$ is continuous and strictly decreasing and $w(\theta)$ is continuous. Because $S_G(m) = W(m)$, we define its inverse

$$H^{-1}(x) = \begin{cases} = W^{-1}\left(\frac{V_I}{x} + L\right) & \text{if } x \geq \frac{V_I}{1-L} \\ = \infty & \text{if } x < \frac{V_I}{1-L} \end{cases} \tag{A.20}$$

The second term in (A.20) is necessary because the height of the bribe must be at least $\frac{V_I}{1-L}$ for the government to win. We first rewrite equations (3.7a)–(3.7b) from Section 3.2.3.

$$T_G(m) = \begin{cases} H(m)S_G(m) - \int_{-\frac{1}{2}}^{m} v(\theta)w(\theta)d\theta & \text{if } m < H^{-1}\left(v(-\frac{1}{2})\right) \quad \text{(A.21a)} \\ H(m)\left(S_G(m) - W\left(v^{-1}\left(H(m)\right)\right)\right) - \int_{v^{-1}(H(m))}^{m} v(\theta)w(\theta)d\theta & \text{if } m \geq H^{-1}\left(v(-\frac{1}{2})\right) \quad \text{(A.21b)} \end{cases}$$

Recall that by definition $S_G(m) = W(m)$. First $T_G(m)$ is continuous on $(\max(\theta_N, \theta_L), \frac{1}{2})$ for each separate region defined by (A.21a) and (A.21b). This follows because $v(\theta)$ is continuous strictly decreasing, and thus $v^{-1}(\theta)$ is strictly increasing and continuous, $w(\theta)$ is continuous, and hence so are $W(m)$, $S_G(m)$, and $H(m)$. We can see that $T_G(m)$ is also continuous at the boundary $m = H^{-1}\left(v(-\frac{1}{2})\right)$ by substituting this value into both equation (A.21a) and (A.21b) and verifying they produce the same result. Next, we differentiate $T_G(m)$ to analyze its behavior and determine where its minimum will be.

$$T'_G(m) = \begin{cases} w(m)\left(-\frac{H(m)}{(S_G(m) - L)}L - v(m)\right) & \text{if } m < H^{-1}\left(v(-\frac{1}{2})\right) \quad \text{(A.22a)} \\ w(m)\left(-\frac{H(m)}{(S_G(m) - L)}\left(L - W\left(v^{-1}\left(H(m)\right)\right)\right) - v(m)\right) & \text{if } m \geq H^{-1}\left(v(-\frac{1}{2})\right) \quad \text{(A.22b)} \end{cases}$$

This follows through straightforward (though a bit tedious) differentiation. To compute (A.22a)–(A.22b) we use the fact that $S_G(m) = W(m)$ and hence $S'_G(m) = w(m)$. Furthermore,

$$\begin{aligned} H'(m) &= \frac{d}{dm}\left(\frac{V_I}{S_G(m) - L}\right) && \text{(A.23a)} \\ &= -\frac{V_I}{(S_G(m) - L)^2}S'_G(m) && \text{(A.23b)} \\ &= -\frac{H(m)}{S_G(m) - L}w(m) && \text{(A.23c)} \end{aligned}$$

The derivative $T'_G(m)$ defined by (A.22a)–(A.22b) is continuous by continuity of $v(\cdot)$, $w(\cdot)$, $v^{-1}(\cdot)$, $W(\cdot)$ and $H(\cdot)$. Furthermore if we substitute the boundary value $m = H^{-1}\left(v(-\frac{1}{2})\right)$ into both (A.22a) and (A.22b) the values coincide and so the derivative is well defined at the boundary value between the flooded and non-flooded coalition cases.

We next show that there is at most one local minimum of $T_G(m)$. The function $T_G(m)$ is

not concave because of the weight function $w(\theta)$, however we can show that there exists an m^* such that $T_G'(m) < 0$ for $m < m^*$ and $T_G'(m) > 0$ for $m > m^*$ and thus there is a local minimum at m^*. Therefore, the government needs to only consider $m = m^*$ or the two endpoints $m = \max(\theta_N, \theta_L)$ and $m = \frac{1}{2}$.

To do this, let us first consider the flooded coalition case by examining the term in parentheses in equation (A.22a):

$$f(m) = -\frac{H(m)}{(S_G(m) - L)}L - v(m) \tag{A.24}$$

Because $m \geq \theta_N$, it follows that $-v(m) \geq 0$ for all feasible m. Thus, the first term of (A.24) is negative and the second term is nonegative (and will be positive for all $m > \theta_N$ by the strict monotonicity of $v(\theta)$). We will next argue that $f(m)$ is increasing in m. Once we establish this, it follows that since $f(m)$ is increasing in m there will be at most one m^* such that $f(m) < 0$ for $m < m^*$ and $f(m) > 0$ for $m > m^*$. It is possible that $f(m)$ is either positive or negative for all m in our region of interest. Since $w(m) \geq 0$, the sign of $f(m)$ is the sign of $T_G'(m)$ and the value m^* such that $f(m^*) = 0$ (if it exists) is a local minimizer of $T_G(m)$. We need not consider roots of $w(\cdot)$, because they correspond to saddle points. Thus, $T_G(m)$ will either be nonincreasing, nondecreasing, or have a unique local minimum on $m \in [\max(\theta_N, \theta_L), H^{-1}\left(v(-\frac{1}{2})\right)]$. To see that $f(m)$ is increasing, we define $m_1 < m_2$ such that $m_1, m_2 < H^{-1}\left(v(-\frac{1}{2})\right)$. By monotonicity of $v(m)$, $-v(m_2) > -v(m_1)$ and because $H(m)$ is decreasing in m and $S_G(m)$ is increasing in m we have $-\frac{H(m_2)}{(S_G(m_2)-L)}L \geq -\frac{H(m_1)}{(S_G(m_1)-L)}L$. Therefore,

$$f(m_1) = -\frac{H(m_1)}{(S_G(m_1) - L)}L - v(m_1) < -\frac{H(m_2)}{(S_G(m_2) - L)}L - v(m_2) = f(m_2), \tag{A.25}$$

and $f(m)$ is increasing. We can perform the exact same analysis on the term in parentheses in equation (A.22b):

$$r(m) = -\frac{H(m)}{(S_G(m) - L)}\left(L - W\left(v^{-1}\left(H(m)\right)\right)\right) - v(m), \tag{A.26}$$

and show that $r(m)$ is also increasing in m. Thus $T_G(m)$ will either be increasing, decreasing, or have a unique local minimum on $m \in [H^{-1}\left(v(-\frac{1}{2})\right), \frac{1}{2}]$.

We next argue it is impossible for $T_G(m)$ to have two local minima on $[\max(\theta_N, \theta_L), \frac{1}{2}]$: one on $[\max(\theta_N, \theta_L), H^{-1}\left(v(-\frac{1}{2})\right)]$ and one on $[H^{-1}\left(v(-\frac{1}{2})\right), \frac{1}{2}]$. This follows from inspection of

equations (A.24) and (A.26). We can rewrite $r(m)$ in terms of $f(m)$ and some positive quantity

$$r(m) = f(m) + \frac{H(m)}{(S_G(m) - L)} W\left(v^{-1}\left(H(m)\right)\right) \tag{A.27}$$

This implies that if there is a local minimum $m^* \in [\max(\theta_N, \theta_L), H^{-1}\left(v(-\frac{1}{2})\right)]$ such that $f(m^*) = 0$, then $T'_G(m) > 0$ for $m > H^{-1}\left(v(-\frac{1}{2})\right)$ by the monotonicity of $f(m)$ and $r(m)$ and equation (A.27). A similar observation can be made that if there is a local minimum $m^* \in [H^{-1}\left(v(-\frac{1}{2})\right), \frac{1}{2}]$, then $T'_G(m) < 0$ for $m < H^{-1}\left(v(-\frac{1}{2})\right)$.

We have shown that there is at most one local minimum to $T_G(m)$. We now rule out the lower endpoint $m = \max(\theta_N, \theta_L)$ as a potential optimal value for the government. Substituting $m = \theta_N$ into equation (A.22a) results in $T'_G(m) < 0$ because $v(\theta_N) = 0$. Evaluating $\lim_{m \downarrow \theta_L} T'_G(m)$ shows that $T'_G(m) < 0$ in a neighborhood of θ_L because the first term in equation (A.22a) blows up.

Therefore, we are left with two options for the optimal cost minimizing choice m^* for the government. Either m^* corresponds to the unique local minimizer of $T_G(\theta)$, or $m^* = \frac{1}{2}$. The latter case corresponds to Case 3 of Proposition 5 in Section 3.2.3. If the optimal solution is the local minimum, then it will either be the root of $f(m)$ in equation (A.24) (which corresponds to Case 2 of Proposition 5 in Section 3.2.3), or it will be the root of $r(m)$ in equation (A.26) (which corresponds to Case 1 of Proposition 5 in Section 3.2.3).

□

A.4.4 Excess Support

In this section, we prove Proposition 6 in Section 3.2.3. If the government wins, it will receive more than the minimal amount of support then it requires for victory.

Proof. This proposition is only interesting if $\theta_L \geq \theta_N$. As Proposition 3 of Section 3.2.3 states, the optimal strategy of the government is a leveling strategy such that the government will receive support on some interval $[\frac{1}{2}, m]$ for $m \geq \theta_N$. Thus if $\theta_L < \theta_N$, if the government wins $S_G > L$. Now consider $\theta_L \geq \theta_N$. In this case let us assume that the government pays a finite leveling bribe $b_G(\theta)$ such that $v(\theta) + b_G(\theta) = a$ and $S_G = L + \epsilon$. The insurgents only need to buy a fraction ϵ of the government's supporters to win. By Corollary 2, the insurgents will bribe the least costly individuals $\theta \in [\theta_L, m]$, where by construction $\int_{[\theta_L, m]} w(\theta) d\theta = \epsilon$. The

cost to the insurgents to counter-bribe these individuals and win is $\int_{[\theta_L,m]} \tilde{v}(\theta)w(\theta)d\theta = a\epsilon$. For a fixed a, $\lim_{\epsilon\to 0} a\epsilon = 0$ and condition (A.6c) of Proposition 2 will eventually hold for some small ϵ and the insurgents will win. Therefore, in order for the government to ensure victory $a \to \infty$ as $\epsilon \to 0$. However, the government has the constraint V_G, and so for large enough a, the government would concede rather than implementing the bribe to win. Thus, for ϵ small enough, the government cannot prevent an insurgent victory. Therefore, if the government wins, they cannot have an excess support arbitrarily close to 0. □

A.4.5 How Government's Strategy Varies With Parameters

To prove cases 1–3 of Proposition 7 in Section 3.2.5 we first prove the following Lemma

Lemma 4. *The bribe parameter m^* corresponding to the upper bound of the set B_G has the following properties.*

1. *m^* is nondecreasing in V_I.*
2. *m^* is nondecreasing in L.*
3. *m^* is nondecreasing if the population's preferences shift from $v_1(\theta)$ to $v_2(\theta)$ and $|v_2(\theta)| < |v_1(\theta)|$.*

Proof. We examine each case separately.

Case 1:

To see that m^* is nondecreasing in V_I, we examine the derivative of the cost function in equations (A.22a)–(A.22b). Define $V_I^1 < V_I^2$ and m_1^* the optimal m associated with V_I^1 and m_2^* the analagous value for V_I^2. Let us redefine $T_G'(m)$ in equations (A.22a)–(A.22b) as $T_G'(m, V_I)$ to denote the dependence of the cost on V_I. Inspection of (A.22a)–(A.22b) reveals that $T_G'(m_1^*, V_I^2) < T_G'(m_1^*, V_I^1)$ (the relationship holds for any m). This holds because the positive terms in (A.22a)–(A.22b) do not depend upon V_I, but the negative terms grow in absolute value because $H(m)$ increases with V_I. Because m_2^* is either a unique local minimum or $m_2^* = \frac{1}{2}$, if $T_G'(m_1^*, V_I^2) < T_G'(m_1^*, V_I^1)$, then $m_2^* \geq m_1^*$.

Case 2:

Very similar logic shows that m^* is nondecreasing in L. We define $L^1 < L^2$, m_1^*, m_2^* and

$T'_G(m, L)$ in an analogous fashion as we did with V_I above. Inspection of (A.22a)–(A.22b) reveals that $T'_G(m_1^*, L^2) < T'_G(m_1^*, L^1)$. This holds because $H(m)$ increases with L. Thus as with V_I, $m_2^* \geq m_1^*$.

Case 3:
Next we define $|v_2(\theta)| \leq |v_1(\theta)|$, so that the preferences of population 2 are not as strong as those in population 1. As above, we associate m_1^* with population 1 and m_2^* with population 2 and define $T'_G(m, v(\cdot))$ to depend upon the preference function v. Inspection of (A.22a)–(A.22b) reveals that $T'_G(m_1^*, v_2(\cdot)) < T'_G(m_1^*, v_1(\cdot))$. This holds because the positive terms in (A.22a)–(A.22b) decrease and the negative terms can only increase in absolute value. Therefore, just as with V_I and L, we have $m_2^* \geq m_1^*$.

□

Because $S_G(m^*) = W(m^*)$, for a fixed weighting function if m^* is increasing then the support $S_G(m^*)$ must also increase. Therefore, cases 1–3 of Proposition 7 in Section 3.2.5 follow immediately from Lemma 4. Finally, we prove case 4 of Proposition 7 below

Proof. Assume that population 1 has weighting function $W_1(\theta)$ and population 2 has weighting function $W_2(\theta)$ and $W_2(\theta) \geq W_1(\theta)$. We first define m_1^* as the optimal m associated with population 1. Next, define $\hat{m}_2$ such that

$$\hat{m}_2 = \{m \ : \ W_2(m) = W_1(m_1^*)\} \tag{A.28}$$

Because $W_2(\theta) \geq W_1(\theta)$, we have $\hat{m}_2 \leq m_1^*$. We next define $T'_G(m, W(\cdot))$ to denote the explicit dependence of the derivative (A.22a)–(A.22b) upon the weighting function. By assumption of the Proposition, we only need to consider the flooded case in (A.22a). We now argue that $T'_G(\hat{m}_2, W_2(\cdot)) < T'_G(m_1^*, W_1(\cdot))$. This holds because by construction of $\hat{m}_2$ the negative term in (A.22a) is the same for both $T'_G(\hat{m}_2, W_2(\cdot))$ and $T'_G(m_1^*, W_1(\cdot))$. However, the positive term $-v(m)$ is smaller for $T'_G(\hat{m}_2, W_2(\cdot))$ than for $T'_G(m_1^*, W_1(\cdot))$, because $\hat{m}_2 \leq m_1^*$ and $v(\cdot)$ is a decreasing function. Therefore, $\hat{m}_2 < m_2^*$ and $W_2(m_2^*) \geq W_2(\hat{m}_2) = W_1(m_1^*)$. Recall that $S_G(m) = W(m)$, and thus we have completed the proof.

We cannot apply the same analysis to the non-flooded case in (A.22b) because the negative term may decrease in absolute value because of the $W\left(v^{-1}\left(H(m)\right)\right)$ term. $\square$

A.5 Coercion

In this section, we present the analysis on the coercion model presented in Section 3.3.

A.5.1 Max Coercion

Below is the proof of Proposition 8 in Section 3.3

Proof. In this case, if the insurgents set the coercion level to $\alpha = 1$, the cost to the government to bribe the population is (see equation (A.21a))

$$T_G(m) = V_I + H(m)L - q_I(1)\int_{\theta_N}^{m} v(\theta)w(\theta)d\theta \tag{A.29}$$

Since $H(m)$ is nonincreasing in m (see equation (3.6) in Section 3.2.3), we have that

$$H(m) \geq H\left(\frac{1}{2}\right) = \frac{V_I}{1-L} \tag{A.30}$$

Furthermore, because $v(\theta)$ is negative for $\theta \geq \theta_N$ we have

$$-q_I(1)\int_{\theta_N}^{m} v(\theta)w(\theta)d\theta \geq 0 \tag{A.31}$$

Putting together equations (A.29)–(A.31) yields

$$T_G(m) \geq V_I + \frac{V_I}{1-L}L + 0 \tag{A.32a}$$

$$= \frac{V_I}{1-L} \tag{A.32b}$$

Therefore, to win the government will need to execute bribes costing at least $\frac{V_I}{1-L}$, and thus if $\frac{V_I}{1-L} > V_G$ the victory is too costly to achieve. $\square$

A.5.2 Derivative of Cost Function

In this section, we calculate the derivative $T'_G(\alpha)$ presented in equations (3.12a)–(3.12b) in Section 3.3.1. First let us focus on the flooded condition in (3.12a)

$$
\begin{aligned}
T_G'(\alpha) =& H'(\alpha)S_G(m(\alpha)) + H(\alpha)S_G'(m(\alpha))m'(\alpha) - q_G'(\alpha)\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta \\
&- q_I'(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta - q_I(\alpha)v(m(\alpha))w(m(\alpha))m'(\alpha)
\end{aligned} \tag{A.33}
$$

Next, observe that $S_G(x) = W(x)$ so $S_G'(x) = w(x)$ and $H'(\alpha) = -\frac{H(\alpha)}{(S_G(m(\alpha))-L)}S_G'(m(\alpha))m'(\alpha)$ (see equation (3.10) in Section 3.3.1). Substituting these values into equation (A.33) yields

$$
\begin{aligned}
T_G'(\alpha) =& -\frac{H(\alpha)}{(S_G(m(\alpha)) - L)}S_G'(m(\alpha))m'(\alpha)S_G(m(\alpha)) + H(\alpha)w(m(\alpha))m'(\alpha) \\
&- q_G'(\alpha)\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta - q_I'(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta - q_I(\alpha)v(m(\alpha))w(m(\alpha))m'(\alpha) && \text{(A.34a)} \\
=& -\frac{H(\alpha)}{(S_G(m(\alpha)) - L)}w(m(\alpha))m'(\alpha)S_G(m(\alpha)) + H(\alpha)w(m(\alpha))m'(\alpha) \\
&- q_G'(\alpha)\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta - q_I'(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta - q_I(\alpha)v(m(\alpha))w(m(\alpha))m'(\alpha) && \text{(A.34b)} \\
=& w(m(\alpha))m'(\alpha)\left(-\frac{H(\alpha)}{(S_G(m(\alpha)) - L)}S_G(m(\alpha)) + H(\alpha) - q_I(\alpha)v(m(\alpha))\right) \\
&- q_G'(\alpha)\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta - q_I'(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta && \text{(A.34c)} \\
=& w(m(\alpha))m'(\alpha)\left(-\frac{H(\alpha)}{(S_G(m(\alpha)) - L)}L - q_I(\alpha)v(m(\alpha))\right) \\
&- q_G'(\alpha)\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta - q_I'(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta && \text{(A.34d)}
\end{aligned}
$$

However, the first line of equation (A.34d) must be zero:

$$
w(m(\alpha))m'(\alpha)\left(-\frac{H(\alpha)}{(S_G(m(\alpha)) - L)}L - q_I(\alpha)v(m(\alpha))\right) = 0 \tag{A.35}
$$

Equation (A.35) follows because by Proposition (5) in Section 3.2.3 either $m(\alpha) = \frac{1}{2}$ or $m(\alpha)$ is the unique interior minimum satisfying case 2 in Proposition (5). If $m(\alpha) = \frac{1}{2}$ then by Proposition 10 in Section 3.3 $m'(\alpha) = 0$. If $m(\alpha)$ is the interior minimizer, then the term in parentheses must be zero, because $m(\alpha)$ is by definition the value that makes this expression 0 (see case 2 in Proposition 5 or equation (A.22a)). Therefore, equation (A.35) holds and we can

simplify equation (A.34d) to coincide with the derivative in (3.12a) of Section 3.3.1:

$$T'_G(\alpha) = -q'_G(\alpha)\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta - q'_I(\alpha)\int_{\theta_N}^{m(\alpha)} v(\theta)w(\theta)d\theta \tag{A.36}$$

A similarly tedious exercise produces equation (3.12b) in Section 3.3.1.

A.5.3 Situational Awareness Parameter p

Below is the proof of Proposition 11 in Section 3.3.2.

Proof. First, we will rewrite the cost function in equation (3.11a) in terms of α and p:

$$\begin{aligned} T_G(\alpha, p) =& H(\alpha, p)S_G(m(\alpha, p))) \\ & - q_G(\alpha)\int_{-\frac{1}{2}}^{\theta_N} v(\theta)w(\theta)d\theta - (p + (1-p)q_G(\alpha))\int_{\theta_N}^{m(\alpha,p)} v(\theta)w(\theta)d\theta \end{aligned} \tag{A.37}$$

Where $m(\alpha, p)$ is also a function of p. By case 3 of Lemma A.4.5, $m(\alpha, p)$ is nonincreasing in p. As in Section A.5.2, we can compute the derivate of (A.37) to see how the cost varies with p. Nearly identical steps to those in Section A.5.2 yield the following

$$\frac{\partial T_G(\alpha, p)}{\partial p} = -(1 - q_G(\alpha))\int_{\theta_N}^{m(\alpha,p)} v(\theta)w(\theta)d\theta \geq 0 \tag{A.38}$$

The derivative is positive because $0 \leq q_G(\alpha) \leq 1$ and $v(\theta) < 0$ for $\theta > \theta_N$.

□

A.5.4 Optimal Coercion Level

Below is the proof of Proposition 12 in Section 3.3.2.

Proof. Examination of $T'_G(\alpha)$ in equation (3.14) reveals that the sign of the derivative is determined by the term in parentheses because $q'_G(\alpha)$ is nonpositive. The first integral in equation (3.14) does not depend upon α and is a positive quantity. The second integral in equation (3.14) is negative and is nondecreasing in α (in absolute value). This follows because $m(\alpha)$ is nondecreasing in α by Proposition 10 in Section 3.3.1. Thus, the term in parentheses in equation

(3.14) is nonincreasing in α and can have at most one root (which would correspond to a local maximum). Therefore the optimal level of coercion is this local maximum if one exists on $\alpha \in (0, 1)$ or it is one of the boundary points $\alpha = 0$ or $\alpha = 1$.

$\square$

A.5.5 How Coercion Varies with Other Parameters

Below is the proof of Proposition 13 in Section 3.3.2

Proof. This proof follows a very similar structure to Lemma 4

Case 1:

To see that α^* is nonincreasing in V_I we examine the derivative of the cost function in equation (3.14) in Section 3.3.2. Define $V_I^1 < V_I^2$ and α_1^* the optimal coercion associated with V_I^1 and α_2^* the analogous value for V_I^2. Let us redefine $m(\alpha)$ and $T'_G(\alpha)$ to denote the dependence on V_I: $m(\alpha, V_I)$ and $T'_G(\alpha, V_I)$. By case 1 of Lemma 4 we know that $m(\alpha, V_I)$ is nondecreasing in V_I for a fixed α and so $m(\alpha_1^*, V_I^1) \leq m(\alpha_1^*, V_I^2)$. However, this implies that $T'_G(\alpha_1^*, V_I^2) < T'_G(\alpha_1^*, V_I^1)$ (this actually applies for all α, not just α_1^*) because the negative term in equation (3.14) has grown in absolute value. Because α_2^* is either a unique local maximizer or $\alpha_2^* \in \{0, 1\}$ (see of Proposition 12 in Section 3.3.2) if $T'_G(\alpha_1^*, V_I^2) < T'_G(\alpha_1^*, V_I^1)$, then $\alpha_2^* \leq \alpha_1^*$.

Case 2:

Very similar logic shows that α^* is nondecreasing in L. We define $L^1 < L^2$, α_1^*, α_2^* and $m(\alpha, L)$ $T'_G(\alpha, L)$ in an analogous fashion as we did with V_I above. By case 2 of Lemma 4, $m(\alpha, L^1) \leq m(\alpha, L^2)$ (namely α_1^*). Applying this to equation (3.14), we have that $T'_G(\alpha, L^2) < T'_G(\alpha, L^1)$ for all $\alpha \in [0, 1]$. By the same logic as in case 1 with V_I, it must follow that $\alpha_2^* \leq \alpha_1^*$.

Case 3:

By case 3 of Lemma 4, $m(\alpha, p)$ is nonincreasing in p. Thus, the exact same steps can be used as in cases 1 and 2 (except the inequalities are reversed) to show this relationship.

$\square$

A.6 Discrete Model

The analysis for the discrete problem is much more difficult than in the continuous case. In this section, we present potential optimization programs that could be utilized to solve for game strategies. We examine the insurgents' strategy in Section A.6.1 and the government's strategy in A.6.2.

A.6.1 Insurgents' Response

In this section, we show that the insurgents' low cost counter-bribe in (3.17) can be formulated as the solution to a knapsack problem. We then present a recursive algorithm that can solve for the cost of the insurgents' low cost counter-bribe. We first define the discrete set $B_G = \{i \ : \ v_i + b_G(i) > 0\}$ to be the tribes who would support the government in the absence of a counter-bribe by the insurgents. The insurgents observe the set B_G, the bribe $b_G(i)$, and the value S_G (from equation (3.15)) before they offer a counter-bribe. The insurgents need to make a compelling counter-bribe $b_I(i)$ to a subset $B' \in B_G$ such that the government has less than L support (i.e., $\sum_{i \in B \setminus B'} w_i < L$), and the insurgents want to do this at the lowest cost possible. However, we know the value of a compelling counter-bribe to any individual in B_G: $b_I(i) = v_i + b_G(i)$. Thus, we can eliminate the decision variable $b_I(i)$ in (3.17) by limiting ourselves to the set B_G:

$$\min_{x_i} \sum_{i=1}^{N} (v_i + b_G(i)) w_i x_i \tag{A.39a}$$

$$S_G - L \leq \sum_{i \in B_G} w_i x_i \tag{A.39b}$$

$$x_i \in \{0, 1\} \tag{A.39c}$$

The optimization problem in (A.39) is an integer program and can be solved using standard techniques (e.g., branch and bound [60]). However, it can be solved via a recursive algorithm derived from dynamic programming as described in the next proposition

Proposition 15. *We can relabel the tribes such that tribes* $1, 2, \dots, |B_G|$ *are in the set* B_G*, which is the tribes who would support the government in the absence of a counter-bribe by the insurgents. If* F *is the solution to the optimization problem in (A.39), then* $F = f(|B_G|, S_G -$

L, S_G), where the function $f(m, C, D)$ is defined below

$$f(m,C,D) = \begin{cases} 0 & \textit{if } C \leq 0 \quad \text{(A.40a)} \\ v_m + f(m-1, C-w_m, D-w_m) & \textit{if } D - w_n < C \quad \text{(A.40b)} \\ \min\{f(m-1, C, D-w_m), & \\ q \quad v_m + f(m-1, C-w_m, D-w_m)\} & \quad \text{(A.40c)} \end{cases}$$

Proof. In the classic knapsack problem, an individual chooses which of m objects to place in a knapsack of capacity C to maximize the knapsack's value. In the variant examined in (A.39), and individual chooses which of N objects (i.e., tribes) to place in a knapsack to minimize the knapsack's value, provided the knapsack's capacity be at least $S_G - L$. The classic knapsack problem has a well known recursive algorithm associated with it to compute the maximum value [60, 61]. The recursive formula in (A.40a)–(A.40c) is similar, so we will just sketch the details.

Assume an individual has to place some collection of m objects into a knapsack to minimize that knapsack's value, subject to the constraint that the total weight of the knapsack be at least C. Furthermore, the collection of objects m weighs $D \geq C$. We define $f(m, C, D)$ to be the minimum weight associated with this problem and now argue that we can define $f(m, C, D)$ recursively as in (A.40a)–(A.40c). We label the objects 1 through m and we assume to start with all of the objects are in the knapsack. The individual will remove objects one at a time, starting from object m. in such a fashion that when the procedure finishes after the decision for object 1, the knapsack will have the minimum weight $f(m, C, D)$. First, we examine the base case when $C \leq 0$, in which case there is no capacity so the individual will place no objects into the knapsack and $f(m, C, D) = 0$, which corresponds to (A.40a).

We next examine the case where the individual leaves object m in the knapsack. In this case, the value of the knapsack has increased by v_m and the individual needs to fill up at least $C - w_m$ in remaining capacity of the knapsack. As with all dynamic programming problems, we assume the individual will make the optimal decision for objects $m - 1$ through 1 (which weigh a combined $D - w_m$). Thus if the optimal decision is to leave object m in the knapsack, we have the following relationship

$$f(m, C, D) = v_m + f(m-1, C-w_m, D-w_m) \quad \text{if object } m \text{ left in knapsack} \qquad \text{(A.41)}$$

If the optimal decision is to take object m out of the knapsack, then no value is added to the

knapsack, and a capacity C still must be filled with the remaining $m-1$ objects (which weigh $D-w_m$). Thus, if the optimal decision is to remove object m in the knapsack, we have the following relationship

$$f(m,C,D) = f(m-1,C,D-w_m) \quad \text{if object } m \text{ removed from knapsack} \tag{A.42}$$

We start with $D \geq C$, and at every point in the future it must be that weight of the remaining objects be at least as much as the remaining capacity to be filled. Otherwise, the knapsack is infeasible. Thus, if $C > D - w_m$, then we cannot remove object m from the knapsack because it will put us into a position of infeasibility. Therefore, we have case (A.40b), which requires us to leave the object in the knapsack. Finally, to the general case where $C \geq D - w_m$. Because the individual behaves optimally for the remaining $m-1$ objects, the individual's decision for object m should be dictated by the smaller of equation (A.41) and (A.42). This is exactly condition (A.40c). □

Because optimization problem (A.39) is this type of knapsack problem with $m = |B_G|$, $C = S_G - L$ and $D = S_G$, the lowest cost counter-bribe to the insurgents will cost $f(|B_G|, S_G - L, S_G)$. For implementation purposes, one would want to use memoization to compute $f(m,C,D)$ to avoid solving the same subproblems many times.

A.6.2 Government's Strategy

In this section, we derive an optimization problem to solve for the government's low cost winning bribe that is slightly more tractable than the general form presented in equation (3.18).

First let us modify the insurgents optimization problem (3.17)

$$\max_{b_I(i),x_i,Iwins} Iwins \tag{A.43a}$$

$$Iwins\left(L - \sum_{i=1}^{N} w_i(1-x_i)\right) \geq 0 \tag{A.43b}$$

$$\sum_{i}^{N} w_i b_I(i) \leq V_I \tag{A.43c}$$

$$b_I(i) \geq (v_i + b_G(i))x_i \tag{A.43d}$$

$$b_I(i) \geq 0 \tag{A.43e}$$

$$x_i \in \{0,1\} \tag{A.43f}$$

$$Iwins \in \{0,1\} \tag{A.43g}$$

The solution to (A.43) is 1 (i.e., $Iwins = 1$) if and only if the insurgents can make a winning counter-bribe to the government's bribe $b_G(\theta)$. If $Iwins = 1$, then by the constraint (A.43b) the government's support is less than L, and by constraint (A.43c) the insurgents can afford the bribe. Therefore, the insurgents can win. Conversely, if the insurgents win, they make a feasible bribe (which satisfies (A.43c)) and the government has less than L support (which means the term in parentheses of (A.43b) is positive). Therefore, $Iwins = 1$ is a feasible, and so the optimal value is 1. Note that this optimization problem (which has a nonlinear constraint in (A.43b)) reveals whether the insurgents win, but does not specify the lowest cost winning counter-bribe.

From the government's perspective, the insurgents' lowest cost counter-bribe is irrelevant. The only thing that matters to the government is whether the insurgents will make a winning counter-bribe. Thus optimization problem (A.43) is a useful starting point for the analysis of the government. We can incorporate the government into the model by making (A.43) a minimax problem:

$$\min_{b_G(i)} \max_{b_I(i),x_i,Iwins} Iwins \tag{A.44a}$$

$$Iwins\left(L - \sum_{i=1}^{N} w_i(1-x_i)\right) \geq 0 \tag{A.44b}$$

$$\sum_{i}^{N} w_i b_I(i) \leq V_I \tag{A.44c}$$

$$\sum_{i}^{N} w_i b_G(i) \leq V_G \tag{A.44d}$$

$$b_I(i) \geq (v_i + b_G(i))x_i \tag{A.44e}$$

$$b_I(i), b_G(i) \geq 0 \tag{A.44f}$$

$$x_i, Iwins \in \{0,1\} \tag{A.44g}$$

The solution of the optimization problem in to (A.44) is 1 (i.e., $Iwins = 1$) if and only if the insurgents will win the game. We discussed above that if $Iwins$ is 1, the insurgents have a feasible winning counter-bribe. By introducing the $\min$ term in the optimization problem, the government desires for $Iwins = 0$ (i.e., the government wins). If $Iwins = 0$, the insurgents cannot make a winning counter-bribe so the government wins. As above this formulation does not give the government's low cost winning bribe; it just specifies who will win. In many cases, that may suffice.

To determine the government's low cost winning bribe, we propose an iterative decomposition algorithm. The motivation for this algorithm is the Bender's decomposition algorithm [59]. Before formally stating the algorithm, we define the following sub and master problems. In the sub-problem, the insurgents choose their decision variables, assuming the government's decision variables are fixed. For the master problem the roles reverse.

Sub-problem:

$$\max_{b_I(i), x_i, a_1, a_2, Iwins} Iwins \tag{A.45a}$$

$$\left(L - \sum_{i=1}^{N} w_i(1 - x_i)\right) \geq a_1 - a_2 \tag{A.45b}$$

$$\sum_{i}^{N} w_i b_I(i) \leq V_I \tag{A.45c}$$

$$b_I(i) \geq (v_i + b_G(i))x_i \tag{A.45d}$$

$$b_I(i) \geq 0 \tag{A.45e}$$

$$x_i \in \{0, 1\} \tag{A.45f}$$

$$a_1, a_2 \in \{0, 1\} \tag{A.45g}$$

$$Iwins = a_1 + a_2 - 1 \tag{A.45h}$$

Notice that this sub-problem is just a slight modification of A.43. We introduce the auxiliary variables a_1 and a_2 to eliminate the non-linearity in A.43b. We will solve the subproblem for many iterations, and after each solve we will pass the optimal value of x_i to the master problem algorithm. We define x_i^k as the optimal x_i to the sub-problem at iteration k. The master problem is slightly more complicated than the sub-problem. We first present it, then discuss its components.

Master Problem:

$$\min_{b_G(i),y_i} \quad \sum_{i=1}^{N} w_i b_G(i) \tag{A.46a}$$

$$\text{st:} \quad \sum_{i=1}^{N} w_i(b_G(i)+v_i)x_i^k > V_I + \sum_{i\in I} w_i v_i x_i^k(1-y_i), \quad \forall\, k=1,2,3\ldots\ell \tag{A.46b}$$

$$b_G(i) \geq -v_i y_i \quad \forall\, i \in I \tag{A.46c}$$

$$b_G(i) \leq M y_i \tag{A.46d}$$

$$b_G(i) \geq 0 \quad \forall\, i \leq N, \tag{A.46e}$$

$$y_i \in \{0,1\} \tag{A.46f}$$

Where:

- y_i is a binary variable that represents whether the government offers a bribe to tribe i.
- the set I represents tribes that have negative preferences for the government (i.e., $v_i < 0$).
- M is a large number to ensure that $b_G(i)$ has no upper bound in a practical sense. For example $M = \frac{\max_i |v_i| + V_I}{\min_i w_i}$
- k is the iteration number of the algorithm.

Constraint (A.46b) states that for any set of tribes that will support the insurgents (i.e., $x_i^k = 1$), the government must offer a bribe so that the cost to the insurgents is at least V_I. This constraint can be more clearly written as

$$\sum_{i\,:\,b_G(i)+v_i>0} w_i(b_G(i)+v_i)x_i^k > V_I. \tag{A.47}$$

Adding the axillary variable y_i to ensure linearity results in the form in (A.46b). Constraint (A.46c) specifies that if the government makes a bribe to a pro-insurgent tribe, it must be sufficient to gain support from that tribe. Constraint (A.46d) creates the dependency between a positive government bribe and $y_i = 1$. Without this constraint. the government would set $y_i = 0$ for all i.

After the sub-problem solves for an optimal x_i, that vector is passed into the master problem in the form of x_i^k, which creates an additional constraint to (A.46b). The solution to the master problem with this new constraint produces a government bribe that will defeat every potential insurgent counter-bribe produced by the sub-problem up to that point in the algorithm. This government bribe is then an input to the sub-problem on the next iteration. The algorithm proceeds until the solution to the subproblem is $Iwins = 0$ (i.e., the insurgents cannot produce a winning counter-bribe for the updated government bribe). Convergence is guaranteed to occur, because there is no practical upper bound to the government bribe $b_G(i)$, and so eventually it will be too costly for the insurgents to counter. Below we summarize the algorithm.

Minimum Cost Algorithm:

1. Set $b_G(i) = 0$, $k = 1$ and solve the sub-problem. Store the optimal value of x_i.
   - If $Iwins = 0$ stop, set the government's optimal bribe to $b_G^*(i) = 0$. (Similar to Case 1 in Section 3.2.4 of the continuous case).
   - else go to step 2.
2. while $Iwins = 1$
   - Pass the optimal solution x_i of the sub-problem to the master problem as the input x_i^k. This adds a constraint to the master problem in (A.46b).
   - Solve the master problem with this additional constraint.
   - Pass the optimal solution $b_G(i)$ of the master problem as input to the sub-problem.
   - Solve the sub-problem..
   - Increment k by one.
   - Check the value of $Iwins$.

When this algorithm terminates, the last optimal solution $b_G(i)$ to the master problem is the government's lowest cost bribe.

We conclude with the caveat that we have not had sufficient time to properly vet this algorithm. It is based on the principles of the Bender's decomposition algorithm [59], produces the correct answer for small verifiable test cases, and appears to produce reasonable solutions for larger problems. However, a more rigorous analysis of this algorithm must be undertaken.

THIS PAGE INTENTIONALLY LEFT BLANK

Initial Distribution List

1. Defense Technical Information Center
   Ft. Belvoir, Virginia

2. Dudly Knox Library
   Naval Postgraduate School
   Monterey, California

3. Professor Michael Atkinson
   Operations Research Department
   Naval Postgraduate School
   Monterey, California

4. Professor Roberto Szechtman
   Operations Research Department
   Naval Postgraduate School
   Monterey, California

5. Barış Yıldız
   Genel Kurmay Başkanlığı
   Ankara, TURKEY

6. Kara Harp Okulu Kutuphanesi
   Bakanlıklar - 06100
   Ankara, TURKEY

7. Savunma Bilimleri Enstitusu
   Kara Harp Okulu
   Bakanlıklar, Ankara, TURKEY

8. Bilkent University Library
   Bilkent Universitesi
   Ankara, TURKEY